Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 6269–6284
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

The chromatic number of 5-valent circulants

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Received 4 December 2006; received in revised form 28 November 2007; accepted 29 November 2007
Available online 8 January 2008

Abstract

A circulant $C(n; S)$ with connection set $S = \{a_1, a_2, \ldots, a_m\}$ is the graph with vertex set $\mathbb{Z}_n$, the cyclic group of order n, and edge set $E = \{\{i, j\} : |i - j| \in S\}$. The chromatic number of connected circulants of degree at most four has been previously determined completely by Heuberger [C. Heuberger, On planarity and colorability of circulant graphs, Discrete Math. 268 (2003) 153–169]. In this paper, we determine completely the chromatic number of connected circulants $C(n; a, b, n/2)$ of degree 5. The methods used are essentially extensions of Heuberger's method but the formulae developed are much more complex.

Keywords: Chromatic number; Circulant

1. Introduction

Let n and $1 \le a_1 < a_2 < \cdots < a_m \le \lfloor n/2 \rfloor$ be positive integers. A graph $G = (V, E)$ is called a *circulant* of order n if $V = \{0, 1, \ldots, n-1\}$, $E = \{\{i, i + a_j \pmod n\} : 0 \le i \le n-1, 1 \le j \le m\}$, and is denoted by $C(n; a_1, a_2, \ldots, a_m)$.

The chromatic number of circulants of degree ≤ 4 has been completely determined in [2]. In this paper we deal with connected 5-valent circulants. Therefore $n \ge 6$ is even, $m = 3$ and $a_3 = n/2$. We use the notation $G = C(n; a, b, n/2)$, where $1 \le a, b < n/2$ and $a \ne b$. Notice that G is connected iff $\gcd(a, b, n/2) = 1$.

The following result completely determines the chromatic number of connected 5-valent circulants.

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doi:10.1016/j.disc.2007.11.065

Theorem 1. *Let $n \geq 6$ be even and $G = C(n; a, b, n/2)$ be a connected circulant. Let $g = \gcd(a, b, n)$. Then*

$$\chi(G) = \begin{cases} 6 & \text{if } G = C(6; 1, 2, 3) \\ 5 & \text{if } G = C(10; 2, 4, 5) \\ 5 & \text{if } 4 \nmid n \text{ and } a + b = n/2 \\ 4 & \text{if } 4 \mid n \text{ and } a + b = n/2 \\ 4 & \text{if } b = n/4 \text{ or } a = n/4 \\ 4 & \text{if } (g = 1 \text{ or } 6 \nmid n), (b \equiv \pm 2a \pmod{n} \text{ or } a \equiv \pm 2b \pmod{n}), \\ & G \neq C(10; 2, 4, 5) \text{ and } G \neq C(6; 1, 2, 3) \\ 4 & \text{if } n = 16, 24, 28 \text{ and } (b \equiv \pm 5a \pmod{n} \text{ or } a \equiv \pm 5b \pmod{n}) \\ 4 & \text{if } n = 26, g = 2 \text{ and } (b \equiv \pm 5a \pmod{n} \text{ or } a \equiv \pm 5b \pmod{n}) \\ 4 & \text{if } n = 20 \text{ and } (b \equiv \pm 6a \pmod{n} \text{ or } a \equiv \pm 6b \pmod{n}) \\ 4 & \text{if } n = 22, g = 1 \text{ and } (b \equiv \pm 8a \pmod{n} \text{ or } a \equiv \pm 8b \pmod{n}) \\ 4 & \text{if } n = 28 \text{ and } (b \equiv \pm 8a \pmod{n} \text{ or } a \equiv \pm 8b \pmod{n}) \\ 4 & \text{if } n = 40 \text{ and } (b \equiv \pm 11a \pmod{n} \text{ or } a \equiv \pm 11b \pmod{n}) \\ 2 & \text{if } 4 \nmid n \text{ and both } a, b \text{ are odd} \\ 3 & \text{otherwise.} \end{cases}$$

2. Preliminaries

Given a graph $G = (V, E)$ and a group $H \leq \operatorname{Aut} G$ we define a quotient graph $\bar{G} = G/H = (\bar{V}, \bar{E})$ by taking $\bar{V}$ to be the orbits of the action of H on V and joining orbits $\bar{v}$ to $\bar{u}$ if and only if there exist adjacent vertices $v \in \bar{v}$ and $u \in \bar{u}$. If every orbit is formed by an independent set of vertices then the mapping $u \mapsto \bar{u}$ defines a graph homomorphism $G \to \bar{G}$ and clearly a k-coloring of $\bar{G}$ lifts to a k-coloring of G.

Notice that, since $G = C(n; a, b, n/2)$ is connected, integers a, b and $n/2$ generate the cyclic group of order n. Moreover, $g = \gcd(a, b, n) = 1$ or $g = 2$. Thus a subgraph $C(n; a, b)$ of G is either connected or it has exactly two connectivity components. We say that two circulants $G = C(n; a, b, \frac{n}{2})$ and $\bar{G} = C(n; \bar{a}, \bar{b}, \frac{n}{2})$ of order n are *multiplier-isomorphic* if there is an isomorphism of the form $V(G) \ni i \mapsto mi \in V(\bar{G})$ for some multiplier $m \in \mathbb{Z}_n$. This implies the following claim.

Claim 2. *A connected 5-valent circulant $C(n; a, b, \frac{n}{2})$ is multiplier-isomorphic with a circulant $C(n; \bar{a}, \bar{b}, \frac{n}{2})$, where either $\gcd(\bar{a}, \bar{b}) = 1$ or $\gcd(\bar{a}, \bar{b}) = 2$; in the latter case $n \equiv 2 \bmod 4$.* □

Moreover, if $\gcd(a, n) = 1$ (or $\gcd(b, n) = 1$) we can easily transform G to a circulant of the form $C(n; 1, \bar{b}, \frac{n}{2})$.

Claim 3. $\gcd(a, n) = 1$ *then a circulant $C(n; a, b, \frac{n}{2})$ is multiplier-isomorphic with a circulant $C(n; 1, \bar{b}, \frac{n}{2})$, where $\bar{b} = a^{-1}b \pmod{n}$.* □

From now on we assume that G is not K_6. By Brooks' theorem [1], we get an upper bound.

Claim 4. $\chi(G) \leq 5$. □

If a, b and $\frac{n}{2}$ are odd integers then $G = C(n; a, b, \frac{n}{2})$ is a bipartite graph and immediately we get the following.

Lemma 5. *Let $n \equiv 2 \pmod{4}$ and a, b be odd. Then $G = C(n; a, b, \frac{n}{2})$ is 2-colorable.* □

3. Case $a = 1$

Lemma 6. *Let $n \equiv 2 \pmod{4}$, b is even and $4 \leq b < \frac{n}{4}$. Then $G = C(n; 1, b, \frac{n}{2})$ is 3-colorable.*

Proof. Let $n = 2m$. Then m is odd. Denote by x and r unique integers satisfying the equality $m = bx + r$, where $0 \leq r < b$. Hence $x \geq 2$, r is odd and $1 \leq r \leq b - 1$. Given x and r, we define a 3-coloring of G by specifying a coloring sequence $w_{x,r}$ of length n over the alphabet $\{B, G, R\}$.

I. x is even. Let

$w_{x,r} = [(BG)^{\frac{b}{2}}(RB)^{\frac{b-2}{2}}GR]^{\frac{x-2}{2}}(BG)^{\frac{b}{2}}(RB)^{\frac{b}{2}}(GR)^{\frac{r-1}{2}}G.$
$[R(BG)^{\frac{b-2}{2}}R(BG)^{\frac{b}{2}}]^{\frac{x-2}{2}}R(BG)^{\frac{b-2}{2}}R(BR)^{\frac{b-2}{2}}BG(RG)^{\frac{r-1}{2}}R.$

II. x is odd. Let
$w_{x,r} = [BG(RG)^{\frac{b-4}{2}}RBGR(BR)^{\frac{b-4}{2}}BG]^{\frac{x-1}{2}}RB(GB)^{\frac{b-4}{2}}GR(BR)^{\frac{r-1}{2}}B.$
$[(GR)^{\frac{b-2}{2}}BG(RB)^{\frac{b-2}{2}}GR]^{\frac{x-1}{2}}(BG)^{\frac{b}{2}}(RB)^{\frac{r-1}{2}}R.$

To verify the correctness of the colorings prescribed above we have to compare sequences $w_{x,r}$, $w_{x,r}\rho^b$, $w_{x,r}\rho^m$, where $w_{x,r}\rho^j$ is the cyclic shift of $w_{x,r}$ by j terms to the right. Note that any two consecutive terms as well as the first and last terms in $w_{x,r}$ are distinct. We have to check that the ith term of $w_{x,r}$ differs from the corresponding ith terms of $w_{x,r}\rho^b$ and $w_{x,r}\rho^m$, for every $i = 1, 2, \ldots, n$.

I. Let $s = \frac{b-2}{2}$, $t = \frac{x-2}{2}$, $z = \frac{r-1}{2}$. Then $b = 2s+2$, $r = 2z+1$ and $x = 2t+2$. Note that $s \ge 1$, $t \ge 0$, $z \ge 0$ and $s \ge z$. For $s > z$ we get

$$\begin{aligned}
w_{x,r} =\ & BG(BG)^{s-z-1}(BG)^z BG[(RB)^s GRBG(BG)^{s-1}BG]^t(RB)^s RB \quad (GR)^z G\\
& R(BG)^{s-z}(BG)^z R \quad [B(GB)^{s-1}GBGR(BG)^s R]^t BR(BR)^{s-1}BG(RG)^z R\\
w_{x,r}\rho^b =\ & RB(RB)^{s-z-1}(GR)^z GR\,[(BG)^s BGRB(RB)^{s-1}GR]^t(BG)^s BG \quad (RB)^z R\\
& B(RB)^{s-z}(GR)^z G \quad [R(BG)^{s-1}BGRB(GB)^s G]^t RB(GB)^{s-1}GR(BR)^z B\\
w_{x,r}\rho^m =\ & RB(GB)^{s-z-1}(GB)^z GR\,[(BG)^s BGRB(GB)^{s-1}GR]^t(BR)^s BG \quad (RG)^z R\\
& B(GB)^{s-z}(GB)^z G \quad [R(BR)^{s-1}BGRB(GB)^s G]^t RB(RB)^{s-1}RB\,(GR)^z G
\end{aligned}$$

while for $s = z$

$$\begin{aligned}
w_{x,r} =\ & B(GB)^s G[(RB)^s GRBG(BG)^{s-1}BG]^t(RB)^s RB \quad (GR)^s G\\
& R(BG)^s R[B(GB)^{s-1}GBGR(BG)^s R]^t BR(BR)^{s-1}BG(RG)^s R\\
w_{x,r}\rho^b =\ & G(RG)^s R[(BG)^s BGRB(RB)^{s-1}GR]^t(BG)^s BG \quad (RB)^s R\\
& B(GR)^s G[R(BG)^{s-1}BGRB(GB)^s G]^t RB(GB)^{s-1}GR(BR)^s B\\
w_{x,r}\rho^m =\ & R(BG)^s R[(BG)^s BGRB(GB)^{s-1}GR]^t(BR)^s BG \quad (RG)^s R\\
& B(GB)^s G[R(BR)^{s-1}BGRB(GB)^s G]^t RB(RB)^{s-1}RB\,(GR)^s G.
\end{aligned}$$

II. Let $s = \frac{b-4}{2}$, $t = \frac{x-3}{2}$, $z = \frac{r-1}{2}$. Then $b = 2s+4$, $r = 2z+1$ and $x = 2t+3$. Note that $s \ge 0$, $t \ge 0$, $z \ge 0$ and $s \ge z-1$. For $s \ge z$ we get

$$\begin{aligned}
w_{x,r} =\ & BG(RG)^{s-z}(RG)^z RBGR(BR)^s BG[BG(RG)^s RBGR(BR)^s BG]^t RB(GB)^s GR\,B(RB)^z\\
& GR(GR)^{s-z}(GR)^z BGRB(RB)^s GR\,[GR(GR)^s BGRB(RB)^s GR]^t BG(BG)^s BGR(BR)^z\\
w_{x,r}\rho^b =\ & GB(GB)^{s-z}GR(BR)^z BG(RG)^s RB[GR(BR)^s BGBG(RG)^s RB]^t GR(BR)^s BG\,R(BG)^z\\
& BG(BG)^{s-z}(RB)^z RB\,GR(GR)^s BG[RB(RB)^s GRGR(GR)^s BG]^t RB(RB)^s GR\,B(GB)^z\\
w_{x,r}\rho^m =\ & GR(GR)^{s-z}(GR)^z BGRB(RB)^s GR\,[GR(GR)^s BGRB(RB)^s GR]^t BG(BG)^s BGR(BR)^z\\
& BG(RG)^{s-z}(RG)^z RBGR(BR)^s BG[BG(RG)^s RBGR(BR)^s BG]^t RB(GB)^s GR\,B(RB)^z
\end{aligned}$$

while for $s = z-1$

$$\begin{aligned}
w_{x,r} =\ & BG(RG)^s RBGR(BR)^s BG[BG(RG)^s RBGR(BR)^s BG]^t RB(GB)^s GR\,B(RB)^{s+1}\\
& GR(GR)^s BGRB(RB)^s GR\,[GR(GR)^s BGRB(RB)^s GR]^t BG(BG)^s BGR(BR)^{s+1}\\
w_{x,r}\rho^b =\ & GR(BR)^s BR\,BG(RG)^s RB[GR(BR)^s BGBG(RG)^s RB]^t GR(BR)^s BG\,R(BG)^{s+1}\\
& RB(RB)^s RB\,GR(GR)^s BG[RB(RB)^s GRGR(GR)^s BG]^t RB(RB)^s GR\,B(GB)^{s+1}\\
w_{x,r}\rho^m =\ & GR(GR)^s BGRB(RB)^s GR\,[GR(GR)^s BGRB(RB)^s GR]^t BG(BG)^s BGR(BR)^{s+1}\\
& BG(RG)^s RBGR(BR)^s BG[BG(RG)^s RBGR(BR)^s BG]^t RB(GB)^s GR\,B(RB)^{s+1}. \quad \square
\end{aligned}$$

Lemma 7. *Let* $n \equiv 0 \pmod 4$ *and* $2 < b \le \frac{n}{6}$. *Then* $G = C(n; 1, b, \frac{n}{2})$ *is* 3-*colorable.*

Proof. Let $n = 2m$. Denote by x and r unique integers satisfying the equality $m = bx + r$, where $0 \le r < b$. Hence $x \ge 3$. Given x and r, we define a 3-coloring of G by specifying a coloring sequence $w_{x,r}$ of length n over the alphabet $\{B, G, R\}$.

I. b is even and x is odd. Then r is even.
Let $w_{x,r} = [GR(BG)^{\frac{b}{2}}(RB)^{\frac{b-2}{2}}]^{\frac{x-1}{2}}(GR)^{\frac{b}{2}}(BG)^{\frac{r}{2}}[BG(RB)^{\frac{b-2}{2}}GR(BG)^{\frac{b-2}{2}}]^{\frac{x-1}{2}}(RB)^{\frac{b}{2}}(GR)^{\frac{r}{2}}.$

II. b is even and x is even. Then r is even.
Let $w_{x,r} = [GR(BG)^{\frac{b-2}{2}} RB(GR)^{\frac{b-2}{2}}]^{\frac{x-2}{2}} BG(RB)^{\frac{b-2}{2}} (GR)^{\frac{b-2}{2}} (BG)^{\frac{r+2}{2}}$ $[(RB)^{\frac{b}{2}} GR(BG)^{\frac{b-2}{2}}]^{\frac{x-2}{2}} RB(GR)^{\frac{b-2}{2}} BG(RB)^{\frac{b-2}{2}} (GR)^{\frac{r}{2}}$.
III. b is odd and x is odd. Then r is odd.
Let $w_{x,r} = [(GR)^{\frac{b-1}{2}} (GB)^{\frac{b+1}{2}}]^{\frac{x-1}{2}} (RB)^{\frac{b+r}{2}} [(RG)^{\frac{b+1}{2}} (BG)^{\frac{b-1}{2}}]^{\frac{x-1}{2}} (BR)^{\frac{b+r}{2}}$.
IV. b is odd and x is even. Then r is even.
Let $w_{x,r} = [(GR)^{\frac{b-1}{2}} (GB)^{\frac{b+1}{2}}]^{\frac{x-2}{2}} (RB)^{\frac{2b+r}{2}} [(RG)^{\frac{b+1}{2}} (BG)^{\frac{b-1}{2}}]^{\frac{x-2}{2}} (BR)^{\frac{2b+r}{2}}$.

Similarly as in the above proof, we verify the correctness of the prescribed colorings by comparing sequences $w_{x,r}$, $w_{x,r}\rho^b$, $w_{x,r}\rho^m$.

I. Let $s = \frac{b-2}{2}$, $t = \frac{x-3}{2}$, $z = \frac{r}{2}$. Then $b = 2s+2$, $r = 2z$ and $x = 2t+3$. Note that $s \geq 1$, $t \geq 0$, $z \geq 0$ and $s \geq z$. We get

$$\begin{aligned}
w_{x,r} =& GR(BG)^{s-z}(BG)^z BG(RB)^s [GR(BG)^s BG(RB)^s]^t GR(GR)^s (BG)^z \\
& BG(RB)^{s-z}(RB)^z GR(BG)^s [BG(RB)^s GR(BG)^s]^t RB(RB)^s (GR)^z \\
w_{x,r}\rho^b =& RB(RB)^{s-z}(GR)^z GR(BG)^s [BG(RB)^s GR(BG)^s]^t BG(RB)^s (GR)^z \\
& GR(GR)^{s-z}(BG)^z BG(RB)^s [GR(BG)^s BG(RB)^s]^t GR(BG)^s (RB)^z \\
w_{x,r}\rho^m =& BG(RB)^{s-z}(RB)^z GR(BG)^s [BG(RB)^s GR(BG)^s]^t RB(RB)^s (GR)^z \\
& GR(BG)^{s-z}(BG)^z BG(RB)^s [GR(BG)^s BG(RB)^s]^t GR(GR)^s (BG)^z.
\end{aligned}$$

II. Let $s = \frac{b-2}{2}$, $t = \frac{x-4}{2}$, $z = \frac{r}{2}$. Then $b = 2s+2$, $r = 2z$ and $x = 2t+4$. Note that $s \geq 1$ and $t \geq 0$. If $0 < z < s$ we get

$$\begin{aligned}
w_{x,r} =& GR(BG)^{s-z}(BG)^z \quad RB(GR)^s [GR(BG)^s RB(GR)^s]^t BG(RB)^s GR(GR)^{s-1} BG(BG)^z \\
& RB\ (RB)^{s-z-1}(RB)^{z+1} GR(BG)^s [RB(RB)^s GR(BG)^s]^t RB(GR)^s BG(RB)^{s-1} RB\, GR(GR)^{z-1} \\
w_{x,r}\rho^b =& RB(RB)^{s-z}(GR)^z \quad GR(BG)^s [RB(GR)^s GR(BG)^s]^t RB(GR)^s BG(RB)^{s-1} RB\, (GR)^z \\
& GR\ (GR)^{s-z-1}(BG)^{z+1} RB(RB)^s [GR(BG)^s RB(RB)^s]^t GR(BG)^s RB(GR)^{s-1} GR\, BG(RB)^{z-1} \\
w_{x,r}\rho^m =& RB(RB)^{s-z}(RB)^z \quad GR(BG)^s [RB(RB)^s GR(BG)^s]^t RB(GR)^s BG(RB)^{s-1} RB\, (GR)^z \\
& GR\ (BG)^{s-z-1}(BG)^{z+1} RB(GR)^s [GR(BG)^s RB(GR)^s]^t BG(RB)^s GR(GR)^{s-1} BGBG(BG)^{z-1}
\end{aligned}$$

but for $z = s$

$$\begin{aligned}
w_{x,r} =& GR(BG)^s RB(GR)^s [GR(BG)^s RB(GR)^s]^t BG(RB)^s GR(GR)^{s-1} BG(BG)^s \\
& RB(RB)^s GR(BG)^s [RB(RB)^s GR(BG)^s]^t RB(GR)^s BG(RB)^{s-1} RB\, GR(GR)^{s-1} \\
w_{x,r}\rho^b =& RB(GR)^s GR(BG)^s [RB(GR)^s GR(BG)^s]^t RB(GR)^s BG(RB)^{s-1} RB\, (GR)^s \\
& BG(BG)^s RB(RB)^s [GR(BG)^s RB(RB)^s]^t GR(BG)^s RB(GR)^{s-1} GR\, BG(RB)^{s-1} \\
w_{x,r}\rho^m =& RB(RB)^s GR(BG)^s [RB(RB)^s GR(BG)^s]^t RB(GR)^s BG(RB)^{s-1} RB\, (GR)^s \\
& GR(BG)^s RB(GR)^s [GR(BG)^s RB(GR)^s]^t BG(RB)^s GR(GR)^{s-1} BGBG(BG)^{s-1}
\end{aligned}$$

and for $z = 0$

$$\begin{aligned}
w_{x,r} =& GR(BG)^s \quad RB(GR)^s [GR(BG)^s RB(GR)^s]^t BG(RB)^s GR(GR)^{s-1} BG \\
& RB\ (RB)^{s-1} RB\, GR(BG)^s [RB(RB)^s GR(BG)^s]^t RB(GR)^s BG(RB)^{s-1} RB \\
w_{x,r}\rho^b =& BG(RB)^s \quad GR(BG)^s [RB(GR)^s GR(BG)^s]^t RB(GR)^s BG(RB)^{s-1} RB \\
& GR\ (GR)^{s-1} BGRB(RB)^s [GR(BG)^s RB(RB)^s]^t GR(BG)^s RB(GR)^{s-1} GR \\
w_{x,r}\rho^m =& RB(RB)^s \quad GR(BG)^s [RB(RB)^s GR(BG)^s]^t RB(GR)^s BG(RB)^{s-1} RB \\
& GR\ (BG)^{s-1} BGRB(GR)^s [GR(BG)^s RB(GR)^s]^t BG(RB)^s GR(GR)^{s-1} BG.
\end{aligned}$$

III. Let $s = \frac{b-1}{2}$, $t = \frac{x-3}{2}$, $z = \frac{r-1}{2}$. Then $b = 2s+1$, $r = 2z+1$ and $x = 2t+3$. Note that $s \geq 1$, $t \geq 0$ and $z \geq 0$. We get

$$\begin{aligned}
w_{x,r} =& (GR)^s GB(GB)^s [(GR)^s GB(GB)^s]^t (RB)^s (RB)^{z+1} \\
& (RG)^s RG(BG)^s [(RG)^s RG(BG)^s]^t (BR)^{s+1} (BR)^z \\
w_{x,r}\rho^b =& (RB)^s RG(RG)^s [(BG)^s BG(RG)^s]^t (BG)^s (BR)^{z+1} \\
& (BR)^s BR(GR)^s [(GB)^s GR(GR)^s]^t (GB)^{s+1} (RB)^z \\
w_{x,r}\rho^m =& (RG)^s RG(BG)^s [(RG)^s RG(BG)^s]^t (BR)^s (BR)^{z+1} \\
& (GR)^s GB(GB)^s [(GR)^s GB(GB)^s]^t (RB)^{s+1} (RB)^z.
\end{aligned}$$

IV. Let $s = \frac{b-1}{2}$, $t = \frac{x-4}{2}$, $z = \frac{r}{2}$. Then $b = 2s+1$, $r = 2z$ and $x = 2t+4$. Note that $s \geq 1$, $t \geq 0$ and $z \geq 0$. We get

$$\begin{aligned} w_{x,r} =& (GR)^s GB(GB)^s[(GR)^s GB(GB)^s]^t (RB)^s (RB)^{s+z+1} \\ & (RG)^s RG(BG)^s[(RG)^s RG(BG)^s]^t (BR)^{s+1}(BR)^{s+z} \\ w_{x,r}\rho^b =& (RB)^s RG(RG)^s[(BG)^s BG(RG)^s]^t (BG)^s (BR)^{s+z+1} \\ & (BR)^s BR(GR)^s[(GB)^s GR(GR)^s]^t (GB)^{s+1}(RB)^{s+z} \\ w_{x,r}\rho^m =& (RG)^s RG(BG)^s[(RG)^s RG(BG)^s]^t (BR)^s (BR)^{s+z+1} \\ & (GR)^s GB(GB)^s[(GR)^s GB(GB)^s]^t (RB)^{s+1}(RB)^{s+z}. \quad \square \end{aligned}$$

Lemma 8. *Let $n \equiv 0 \pmod 4$ and $\frac{n}{6} < b < \frac{n}{4}$. Then $G = C(n; 1, b, \frac{n}{2})$ is 3-colorable except when either $G = C(16; 1, 3, 8)$ or $G = C(24; 1, 5, 12)$ or $G = C(28; 1, 5, 14)$.*

Proof. Let $n = 2m$. Denote by x and r unique integers satisfying the equality $m = bx + r$, where $0 \leq r < b$. Hence $x = 2$, r is even and $2 \leq r \leq b-1$. Given $x = 2$ and r, we define a 3-coloring of G by specifying a coloring sequence $w_{x,r}$ of length n over the alphabet $\{B, G, R\}$.

I. $r \geq 4$. Then $b \leq \frac{n-8}{4}$. Let k and p denote unique integers satisfying the equality $b = kr + p$, where $0 \leq p < r$. Hence $k \geq 1$ and moreover if $k = 1$ then $p > 0$. Consider separately following subcases:

I.A. b is even and k is even. Then p is even. Let

$w_{2,r} = [(GR)^{\frac{r}{2}} BG(RB)^{\frac{r-2}{2}}]^{\frac{k-2}{2}} (GR)^{\frac{r}{2}} (BG)^{\frac{r}{2}} (RB)^{\frac{p}{2}} [(RB)^{\frac{r}{2}} GR(BG)^{\frac{r-2}{2}}]^{\frac{k-2}{2}} (RB)^{\frac{r}{2}} (GR)^{\frac{r}{2}} (BG)^{\frac{r+p}{2}}$
$[BG(RB)^{\frac{r-2}{2}} (GR)^{\frac{r}{2}}]^{\frac{k}{2}} (GR)^{\frac{p}{2}} [GR(BG)^{\frac{r-2}{2}} (RB)^{\frac{r}{2}}]^{\frac{k}{2}} (RB)^{\frac{p+2}{2}} (GR)^{\frac{r-2}{2}}$.

I.B. b is even and k is odd. Then p is even. Let

$w_{2,r} = [(BG)^{\frac{r-2}{2}} RG(RB)^{\frac{r-2}{2}} GR]^{\frac{k-1}{2}} (GR)^{\frac{r-2}{2}} (BG)^{\frac{p+2}{2}} [(RB)^{\frac{r-2}{2}} GR(BG)^{\frac{r-2}{2}} RB]^{\frac{k-1}{2}}$
$(RB)^{\frac{r-2}{2}} (GR)^{\frac{p+2}{2}} (BG)^{\frac{r-2}{2}} RG[(RB)^{\frac{r-2}{2}} GR(BG)^{\frac{r-2}{2}} RB]^{\frac{k-1}{2}} (RB)^{\frac{r-2}{2}} (GR)^{\frac{p+2}{2}}$
$[(BG)^{\frac{r-2}{2}} RG(RB)^{\frac{r-2}{2}} GR]^{\frac{k-1}{2}} (BG)^{\frac{r-2}{2}} RG(BG)^{\frac{p}{2}} (RB)^{\frac{r-2}{2}} GR$.

I.C. b is odd and k is even. Then p is odd. Let

$w_{2,r} = [(BG)^{\frac{r-2}{2}} (RB)^{\frac{r}{2}} GR]^{\frac{k-2}{2}} (BG)^{\frac{r-2}{2}} RB(RG)^{\frac{r-2}{2}} RB(GB)^{\frac{p-1}{2}} G[(RB)^{\frac{r-2}{2}} GR(BG)^{\frac{r-2}{2}} RB]^{\frac{k-2}{2}}$
$(RB)^{\frac{r-2}{2}} GR(BR)^{\frac{r-2}{2}} BG(RG)^{\frac{p-1}{2}} R(BG)^{\frac{r-2}{2}} RB[(RB)^{\frac{r-2}{2}} GR(BG)^{\frac{r-2}{2}} RB]^{\frac{k-2}{2}}$
$(GB)^{\frac{r-2}{2}} GR(BR)^{\frac{r-2}{2}} BG(RG)^{\frac{p-1}{2}} R[(BG)^{\frac{r-2}{2}} (RB)^{\frac{r}{2}} GR]^{\frac{k-2}{2}} (BG)^{\frac{r-2}{2}} RB(GB)^{\frac{r-2}{2}} GR(GB)^{\frac{p-1}{2}} G(RB)^{\frac{r-2}{2}} GR$.

I.D. b is odd and k is odd. Then p is odd. Let

$w_{2,r} = [(RG)^{\frac{r-2}{2}} RB(GR)^{\frac{r-2}{2}} BG]^{\frac{k-1}{2}} RB(GB)^{\frac{r-4}{2}} GR(BR)^{\frac{p-1}{2}} B[(GR)^{\frac{r-2}{2}} BG(RG)^{\frac{r-2}{2}} RB]^{\frac{k-1}{2}}$
$GR(BR)^{\frac{r-4}{2}} BG(RB)^{\frac{p-1}{2}} RBG(RG)^{\frac{r-4}{2}} RB[(GR)^{\frac{r-2}{2}} BG(RG)^{\frac{r-2}{2}} RB]^{\frac{k-1}{2}} GR(BG)^{\frac{r-4}{2}} \omega$
$[(BG)^{\frac{r-2}{2}} RB(GR)^{\frac{r-2}{2}} BG]^{\frac{k-1}{2}} (RB)^{\frac{r-2}{2}} GR(BR)^{\frac{p-1}{2}} B(GR)^{\frac{r-2}{2}} BG$, where $\omega = (RB)^{\frac{p+1}{2}} G$ if $r \geq 6$ and $p \leq r-3$,

$\omega = BG(RB)^{\frac{p-1}{2}} G$ if $p = r-1$, and moreover $\omega = BGR$ if $r = 4$, $p = 1$ and $k > 1$. If $r = 4$, $p = 1$ and $k = 1$ the circulant $G = C(28; 1, 5, 14)$ is not 3-colorable, which was verified by a computer.

II. $r = 2$. Then $b = \frac{n-4}{4}$. Let $b = 3k+1$, where $k \geq 1$, and $w_{2,2} = (BGR)^k G(RBG)^{k+1}(RBG)^k R(BGR)^{k+1}$.

If $b = 3k+3$, where $k \geq 1$, let

$w_{2,2} = (RBG)^{k+1}(BGR)^{k+1} GR(BGR)^k BRB(GRB)^k GBGRB$.

For $b = 3$ the circulant $G = C(16; 1, 3, 8)$ is not 3-colorable (verified by a computer).

If $b = 3k+5$, where $k \geq 1$, let

$w_{2,2} = BG(BGR)^k BRBGR(GRB)^k GBGRBR(BGR)^k GRBGB(GRB)^k RBGRGR$.

For $b = 5$ the circulant $G = C(24; 1, 5, 12)$ is not 3-colorable, as verified by a computer.

Similarly as in the above proof, we verify the correctness of the prescribed colorings by comparing sequences $w_{x,r}$, $w_{x,r}\rho^b$, $w_{x,r}\rho^m$.

I.A. Let $s = \frac{r-2}{2}$, $t = \frac{k-2}{2}$ and $z = \frac{p}{2}$. Then $b = (t+1)(4s+4) + 2z$ and $r = 2s+2$. Note that $s \geq 1$, $t \geq 0$, $z \geq 0$ and $s \geq z$. We get

$$\begin{array}{rll}
w_{2,r} = & [GR(GR)^s BG(RB)^s]^t GR(GR)^s (BG)^{z+1} (BG)^{s-z} (RB)^z & \\
& [RB(RB)^s GR(BG)^s]^t RB(RB)^s GR(GR)^s (BG)^z & BG(BG)^s \\
& [BG(RB)^s GR(GR)^s]^t BG(RB)^s (GR)^{z+1} (GR)^{s-z} (GR)^z & \\
& [GR(BG)^s RB(RB)^s]^t GR(BG)^s RB(RB)^s (RB)^z & RB(GR)^s \\
w_{2,r}\rho^b = & [RB(RB)^s GR(BG)^s]^t RB(RB)^s (RB)^{z+1} (GR)^{s-z} (GR)^z & \\
& [GR(GR)^s BG(RB)^s]^t GR(GR)^s BG(BG)^s (RB)^z & RB(RB)^s \\
& [GR(BG)^s RB(RB)^s]^t GR(GR)^s (BG)^{z+1} (BG)^{s-z} (BG)^z & \\
& [BG(RB)^s GR(GR)^s]^t BG(RB)^s GR(GR)^s (GR)^z & GR(BG)^s \\
w_{2,r}\rho^m = & [BG(RB)^s GR(GR)^s]^t BG(RB)^s (GR)^{z+1} (GR)^{s-z} (GR)^z & \\
& [GR(BG)^s RB(RB)^s]^t GR(BG)^s RB(RB)^s (RB)^z & RB(GR)^s \\
& [GR(GR)^s BG(RB)^s]^t GR(GR)^s (BG)^{z+1} (BG)^{s-z} (RB)^z & \\
& [RB(RB)^s GR(BG)^s]^t RB(RB)^s GR(GR)^s (BG)^z & BG(BG)^s.
\end{array}$$

I.B. Let $s = \frac{r-2}{2}$, $t = \frac{k-1}{2}$ and $z = \frac{p}{2}$. Then $b = t(4s+4) + 2s + 2z + 2$ and $r = 2s+2$. Note that $s \geq 1$, $t \geq 0$, $z \geq 0$, $s \geq z$ and moreover $z + t > 0$. We get

$$\begin{array}{rll}
w_{2,r} = & [(BG)^s RG(RB)^s GR]^t (GR)^z (GR)^{s-z} (BG)^z BG & \\
& [(RB)^s GR(BG)^s RB]^t (RB)^s GR(GR)^z & (BG)^s RG \\
& [(RB)^s GR(BG)^s RB]^t (RB)^z (RB)^{s-z} (GR)^z GR & \\
& [(BG)^s RG(RB)^s GR]^t (BG)^s RG(BG)^z & (RB)^s GR \\
w_{2,r}\rho^b = & [(RB)^s GR(BG)^s RG]^t (BG)^z (RB)^{s-z} (RB)^z GR & \\
& [(BG)^s RG(RB)^s GR]^t (GR)^s BG(BG)^z & (RB)^s GR \\
& [(BG)^s RB(RB)^s GR]^t (GR)^z (BG)^{s-z} (BG)^z RG & \\
& [(RB)^s GR(BG)^s RB]^t (RB)^s GR(GR)^z & (BG)^s RG \\
w_{2,r}\rho^m = & [(RB)^s GR(BG)^s RB]^t (RB)^z (RB)^{s-z} (GR)^z GR & \\
& [(BG)^s RG(RB)^s GR]^t (BG)^s RG(BG)^z & (RB)^s GR \\
& [(BG)^s RG(RB)^s GR]^t (GR)^z (GR)^{s-z} (BG)^z BG & \\
& [(RB)^s GR(BG)^s RB]^t (RB)^s GR(GR)^z & (BG)^s RG.
\end{array}$$

I.C. Let $s = \frac{r-2}{2}$, $t = \frac{k-2}{2}$ and $z = \frac{p-1}{2}$. Then $b = (t+1)(4s+4) + 2z + 1$ and $r = 2s+2$. Note that $s \geq 1$, $t \geq 0$, $z \geq 0$ and $s \geq z$. We get

$$\begin{array}{rll}
w_{2,r} = & [(BG)^s RB(RB)^s GR]^t (BG)^s RB(RG)^z R(GR)^{s-z} (BG)^z BG & \\
& [(RB)^s GR(BG)^s RB]^t (RB)^s GR(BR)^s BG(RG)^z R & (BG)^s RB \\
& [(RB)^s GR(BG)^s RB]^t (GB)^s GR(BR)^z B(RB)^{s-z} (GR)^z GR & \\
& [(BG)^s RB(RB)^s GR]^t (BG)^s RB(GB)^s GR(GB)^z G & (RB)^s GR \\
w_{2,r}\rho^b = & [(RB)^s GR(BG)^s RB]^t (GB)^s GR(GB)^z G(RB)^{s-z} (RB)^z GR & \\
& [(BG)^s RB(RB)^s GR]^t (BG)^s RB(RG)^s RB(GB)^z G & (RB)^s GR \\
& [(BG)^s RB(RB)^s GR]^t (BR)^s BG(RG)^z R(BG)^{s-z} (BG)^z RB & \\
& [(RB)^s GR(BG)^s RB]^t (GB)^s GR(BR)^s BG(RG)^z R & (BG)^s RB \\
w_{2,r}\rho^m = & [(RB)^s GR(BG)^s RB]^t (GB)^s GR(BR)^z B(RB)^{s-z} (GR)^z GR & \\
& [(BG)^s RB(RB)^s GR]^t (BG)^s RB(GB)^s GR(GB)^z G & (RB)^s GR \\
& [(BG)^s RB(RB)^s GR]^t (BG)^s RB(RG)^z R(GR)^{s-z} (BG)^z BG & \\
& [(RB)^s GR(BG)^s RB]^t (RB)^s GR(BR)^s BG(RG)^z R & (BG)^s RB.
\end{array}$$

I.D. Let $s = \frac{r-2}{2}$, $t = \frac{k-1}{2}$ and $z = \frac{p-1}{2}$. Then $b = t(4s+4) + 2s + 2z + 3$ and $r = 2s+2$. Note that $s \geq 1$, $t \geq 0$, $z \geq 0$ and $s \geq z$. For $k \geq 3$ we get

$$
\begin{aligned}
w_{2,r} = {} & [(RG)^s RB(GR)^s BG]^{t-1}(RG)^s RB(GR)^s BG \quad RB(GB)^{s-1}GR(BR)^z B \\
& [(GR)^s BG(RG)^s RB]^{t-1}(GR)^s BG(RG)^s RB \quad GR(BR)^{s-1}BG(RB)^z RBG(RG)^{s-1}RB \\
& [(GR)^s BG(RG)^s RB]^{t-1}(GR)^s BGRG(RG)^{s-1}RBGR(BG)^{s-1}\omega \\
& [(BG)^s RB(GR)^s BG]^{t-1}(BG)^s RB(GR)^s BG \quad (RB)^s GR(BR)^z B \quad (GR)^s BG \\
w_{2,r}\rho^b = {} & [(GR)^s BG(BG)^s RB]^{t-1}(GR)^s BG(RB)^s GR \quad (BR)^z BG(RG)^{s-1}RBG \\
& [(RG)^s RB(GR)^s BG]^{t-1}(RG)^s RB(GR)^s BG \quad RB(GB)^{s-1}GR(BR)^z B(GR)^s BG \\
& [(RG)^s RB(GR)^s BG]^{t-1}(RG)^s RBGR(BR)^{s-1}BG(RB)^{z+1}(GR)^s B \\
& [(GR)^s BG(RG)^s RB]^{t-1}(GR)^s BG(RG)^s RB \quad GR(BG)^{s-1}\omega \quad (BG)^s RB \\
w_{2,r}\rho^m = {} & [(GR)^s BG(RG)^s RB]^{t-1}(GR)^s BG(RG)^s RB \quad GR(BG)^{s-1}\omega \\
& [(BG)^s RB(GR)^s BG]^{t-1}(BG)^s RB(GR)^s BG \quad RB^s GR(BR)^z B \quad (GR)^s BG \\
& [(RG)^s RB(GR)^s BG]^{t-1}(RG)^s RBGR(GR)^{s-1}BGRB(GB)^{s-1}GR(BR)^z B \\
& [(GR)^s BG(RG)^s RB]^{t-1}(GR)^s BG(RG)^s RB \quad GR(BR)^{s-1}BG(RB)^z RBG(RG)^{s-1}RB
\end{aligned}
$$

while for $k = 1$ (then $b = 2s + 2z + 3$)

$$
\begin{aligned}
w_{2,r} = {} & RB(GB)^{s-1}GR(BR)^z BGR(BR)^{s-1}BG(RB)^z RBG(RG)^{s-1}RB \\
& GR(BG)^{s-1}\omega \quad RB(RB)^{s-1}GR(BR)^z BGR(GR)^{s-1}BG \\
w_{2,r}\rho^b = {} & (BR)^z BG(RG)^{s-1}RBGRB(GB)^{s-1}GR(BR)^z BGR(BR)^{s-1}BG \\
& (RB)^{z+1}(GR)^s B \quad GR(BG)^{s-1}\omega \quad RB(RB)^{s-1}GR \\
w_{2,r}\rho^m = {} & GR(BG)^{s-1}\omega \quad RB(RB)^{s-1}GR(BR)^z BGR(GR)^{s-1}BG \\
& RB(GB)^{s-1}GR(BR)^z BGR(BR)^{s-1}BG(RB)^z RBG(RG)^{s-1}RB.
\end{aligned}
$$

II. $r = 2$. For $b = 3k + 1$, where $k \geq 1$, we get

$$
\begin{aligned}
w_{2,2} &= (BGR)^k G(RBG)^k RBG(RBG)^k R(BGR)^k BGR \\
w_{2,2}\rho^b &= (RBG)^k R(BGR)^k GRB(GRB)^k G(RBG)^k RBG \\
w_{2,2}\rho^m &= (RBG)^k R(BGR)^k BGR(BGR)^k G(RBG)^k RBG
\end{aligned}
$$

for $b = 3k + 3$, where $k \geq 1$, we get

$$
\begin{aligned}
w_{2,2} &= (RBG)^{k-1}RBGRBG(BGR)^k BGRGR(BGR)^k BRB(GRB)^k GBGRB \\
w_{2,2}\rho^b &= (BGR)^{k-1}BGBGRB(RBG)^k RBGBG(RBG)^k RGR(BGR)^k BRBGR \\
w_{2,2}\rho^m &= (BGR)^{k-1}BGRBRB(GRB)^k GBGRB(RBG)^k RBG(BGR)^k BGRGR
\end{aligned}
$$

while for $b = 3k + 5$, where $k \geq 1$, we get

$$
\begin{aligned}
w_{2,2} &= BG(BGR)^{k-1}BGRBRBGR(GRB)^k GBGRBR(BGR)^k GRBGB(GRB)^k RBGRGR \\
w_{2,2}\rho^b &= RB(GRB)^{k-1}RBGRGRBG(BGR)^k BRBGRG(RBG)^k BGRBR(BGR)^k GRBGBG \\
w_{2,2}\rho^m &= RB(GRB)^{k-1}GRGRBGB(GRB)^k RBGRGRBG(BGR)^k BRBGR(GRB)^k GBGRB. \quad \square
\end{aligned}
$$

Lemma 9. *Let n be even and $\frac{n}{4} < b < \frac{n}{2} - 1$. Then $G = C(n; 1, b, \frac{n}{2})$ is 3-colorable except when either $G = C(16; 1, 5, 8)$ or $G = C(20; 1, 6, 10)$ or $G = C(22; 1, 8, 11)$ or $G = C(28; 1, 8, 14)$ or $G = C(28; 1, 11, 14)$ or $G = C(40; 1, 11, 20)$.*

Proof. Let $n = 2m$. Denote by x and r unique integers satisfying the equality $m = bx + r$, where $0 \leq r < b$. Hence $x = 1$ and $2 \leq r \leq b-1$. Notice that b and r are of the same parity if $n \equiv 0 \pmod 4$ and b is even, r is odd otherwise. Given b and r, denote by k and p unique integers satisfying the equality $b = kr + p$, where $0 \leq p < r$. Hence $k \geq 1$. Given $x = 1$ and r we define a 3-coloring of G by specifying a coloring sequence $w_{1,r}$ of length n over the alphabet $\{B, G, R\}$.

I. Let $p \geq 3$. Denote by l and q unique integers satisfying the equality $r = lp + q$, where $0 \leq q < p$. Hence $l \geq 1$ and moreover $q \geq 1$ if $l = 1$. Consider separately the following subcases:

I.A. p is even and q is even. Then b and r are even. Let

$w_{1,r} = [(BG)^{\frac{p-2}{2}}RG(RB)^{\frac{p-2}{2}}GR]^{\frac{l}{2}}(GR)^{\frac{p-2}{2}}(BG)^{\frac{q+2}{2}}[[(RB)^{\frac{p-2}{2}}GR(BG)^{\frac{p-2}{2}}RG]^{\frac{l}{2}}(BG)^{\frac{q}{2}}]^k$
$[(RB)^{\frac{p-2}{2}}GR(BG)^{\frac{p-2}{2}}RG]^{\frac{l}{2}}(RB)^{\frac{p-2}{2}}(GR)^{\frac{q+2}{2}}[[(BG)^{\frac{p-2}{2}}RG(RB)^{\frac{p-2}{2}}GR]^{\frac{l}{2}}(GR)^{\frac{q}{2}}]^k$ for even l and

$w_{1,r} = [GR(BG)^{\frac{p-2}{2}}(RB)^{\frac{p-2}{2}}GR]^{\frac{l-1}{2}}(BG)^{\frac{p-2}{2}}RB(GR)^{\frac{p-2}{2}}BG\omega[[(RB)^{\frac{p-2}{2}}GR(BG)^{\frac{p-2}{2}}RB]^{\frac{l-1}{2}}(GR)^{\frac{p-2}{2}}BG\omega]^k$
$[(RB)^{\frac{p-2}{2}}GR(BG)^{\frac{p-2}{2}}RB]^{\frac{l-1}{2}}(GR)^{\frac{p-2}{2}}BG(RB)^{\frac{p-2}{2}}(GR)^{\frac{q+2}{2}}[[(BG)^{\frac{p-2}{2}}(RB)^{\frac{p}{2}}GR]^{\frac{l-1}{2}}(BG)^{\frac{p-2}{2}}RB(GR)^{\frac{q}{2}}]^k$
for odd l, where $\omega = (BG)^{\frac{q}{2}}$ if $l \geq 3$ and $\omega = (BG)^{\frac{q-2}{2}}RB$ otherwise (then $q \geq 2$).

I.B. p is even and q is odd. Then r is odd. Let
$w_{1,r} = [(RG)^{\frac{p-2}{2}}RB(GR)^{\frac{p-2}{2}}BG]^{\frac{l}{2}}RB(GB)^{\frac{p-4}{2}}GR(BR)^{\frac{q-1}{2}}B$
$[[(GR)^{\frac{p-2}{2}}BG(RB)^{\frac{p-2}{2}}GR]^{\frac{l}{2}}(BR)^{\frac{q-1}{2}}B]^k[(GR)^{\frac{p-2}{2}}BG(RB)^{\frac{p-2}{2}}GR]^{\frac{l}{2}}(BG)^{\frac{q+1}{2}}\omega(RB)^{\frac{q-1}{2}}G$
$[[(RB)^{\frac{p-2}{2}}(GR)^{\frac{p}{2}}BG]^{\frac{l-2}{2}}(RB)^{\frac{p-2}{2}}GR(BG)^{\frac{p-2}{2}}RB(GB)^{\frac{q-1}{2}}G]^k$ for even l, where $\omega = (BG)^{\frac{p-q-1}{2}}$ if $q > 1$ and $\omega = (BG)^{\frac{p-4}{2}}RB$ otherwise; moreover
$w_{1,r} = [(RB)^{\frac{p-2}{2}}GR(BR)^{\frac{p-2}{2}}BG]^{\frac{l-1}{2}}(RB)^{\frac{p-2}{2}}(GR)^{\frac{p+2}{2}}(BG)^{\frac{q-1}{2}}B$
$[[(GR)^{\frac{p-2}{2}}BG(RB)^{\frac{p-2}{2}}GR]^{\frac{l-1}{2}}(GR)^{\frac{p-2}{2}}(BG)^{\frac{q+1}{2}}R]^k$
$[(BR)^{\frac{p-2}{2}}BG(RB)^{\frac{p-2}{2}}GR]^{\frac{l-1}{2}}(BR)^{\frac{p-2}{2}}BG(RG)^{\frac{p-2}{2}}RB(GB)^{\frac{q-1}{2}}G$
$[[(RB)^{\frac{p-2}{2}}GR(BR)^{\frac{p-2}{2}}BG]^{\frac{l-1}{2}}(RB)^{\frac{p}{2}}(GB)^{\frac{q-1}{2}}G]^k$ for odd l.

I.C. p is odd and q is even. Then r is odd. Hence l is odd. Let
$w_{1,r} = [(RB)^{\frac{p+1}{2}}(GB)^{\frac{p-1}{2}}]^{\frac{l-1}{2}}(RB)^{\frac{p-1}{2}}RG(BG)^{\frac{p-1}{2}}\omega[[(BG)^{\frac{p-1}{2}}(BR)^{\frac{p+1}{2}}]^{\frac{l-1}{2}}(GR)^{\frac{p-1}{2}}G\omega]^k$
$[(BG)^{\frac{p-1}{2}}(BR)^{\frac{p+1}{2}}]^{\frac{l-1}{2}}(BR)^{\frac{p-1}{2}}(GB)^{\frac{p+q+1}{2}}[[(RB)^{\frac{p+1}{2}}(GB)^{\frac{p-1}{2}}]^{\frac{l-1}{2}}(RG)^{\frac{p-1}{2}}B(GB)^{\frac{q}{2}}]^k$, where $\omega = (BG)^{\frac{q}{2}}$ if $l \geq 3$ and $\omega = (BG)^{\frac{q-2}{2}}BR$ otherwise (then $q \geq 2$).

I.D. p is odd and q is odd. Then r is odd. Hence l is even. Let
$w_{1,r} = (BG)^{\frac{lp-p+1}{2}}(RG)^{\frac{p-1}{2}}(RB)^{\frac{p+q}{2}}[(GB)^{\frac{lp-p-1}{2}}(GR)^{\frac{p+q}{2}}B]^k$
$(GB)^{\frac{lp-p+1}{2}}(GR)^{\frac{p-1}{2}}(BR)^{\frac{p+q}{2}}[(BG)^{\frac{lp-p-1}{2}}(RG)^{\frac{p+q}{2}}R]^k$.

II. Let $p = 2$. Then $b = kr + 2$. Consider separately the following subcases:

II.A. $k = 1$. Let $b = 3i + 4$, where $i \geq 0$, and
$w_{1,r} = (BGR)^i BGBG(RBG)^i RB(GRB)^i RBGR(BGR)^i GR$.
If $b = 3i + 6$, where $i \geq 1$, let
$w_{1,r} = BG(BGR)^i(BG)^2RB(RBG)^i(RB)^2(GRB)^iRB(GR)^2(BGR)^iGR$.
For $b = 6$ the circulant $G = C(20; 1, 6, 10)$ is not 3-colorable (verified by a computer).
If $b = 3i + 11$, where $i \geq 1$, let
$w_{1,r} = GRBG(BGR)^i(BG)^2(RB)^2(GRB)^iGBGRBRBGR(GRB)^{i+1}RB(GR)^2BG(BGR)^{i+1}GRBGB$.
For $b = 5$, $b = 8$ and $b = 11$ the circulants $G = C(16; 1, 5, 8)$ $(\simeq C(16; 1, 3, 8))$, $G = C(28; 1, 8, 14)$ and $G = C(40; 1, 11, 20)$ are not 3-colorable, which was verified by a computer.

II.B. $k \geq 2$ and r is even. Then b is even. Let
$w_{1,r} = (GR)^{\frac{rk-r}{2}}(BG)^{r+1}(RB)^{\frac{rk}{2}}(GR)^{\frac{r+2}{2}}$.

C. $k \geq 2$ and r is odd. Let $r = 2l + 3$, where $l \geq 1$, and
$w_{1,r} = BGR(GR)^l[BGR(BG)^l]^{k-1}BGRBG(BG)^lRBG(RB)^l[GRB(RB)^l]^{k-1}RBGRB(GR)^l$.
If $r = 3$ and $k \geq 4$, let
$w_{1,r} = BGR[GRB]^{k-2}RBGRBGRB[RBG]^{k-2}(BGR)^2BGBGR$.
For $b = 11$ and $r = 3$ (then $k = 3$) as well as for $b = 8$ and $r = 3$ (then $k = 2$) the circulants $G = C(28; 1, 11, 14)$ $(\simeq C(28; 1, 5, 14))$ and $G = C(22; 1, 8, 11)$, respectively, are not 3-colorable (verified by a computer).

III. Let $p = 1$. Then $b = kr + 1$ and r is odd ($r \geq 3$). Let
$w_{1,r} = [(GR)^{\frac{r-1}{2}}B]^kG[(RB)^{\frac{r-1}{2}}G]^{k+1}R(GR)^{\frac{r-1}{2}}B$.

IV. Let $p = 0$. Then $b = kr$ and $k \geq 2$ ($r \geq 2$). Consider separately two subcases:

IV.A. r is even. Then b is even. Let
$w_{1,r} = (BG)^{\frac{kr}{2}}(RB)^{\frac{kr}{2}}(GR)^r$.

IV.B. r is odd. Let
$w_{1,r} = [(GR)^{\frac{r-1}{2}}B]^k[(RB)^{\frac{r-1}{2}}G]^k[(BG)^{\frac{r-1}{2}}R]^2$.

Similarly as in the above proof, we verify the correctness of the prescribed colorings by comparing sequences $w_{x,r}$, $w_{x,r}\rho^b$, $w_{x,r}\rho^m$.

I.A. Let $s = \frac{p-2}{2}$, $t = \lfloor \frac{l}{2} \rfloor$ and $z = \frac{q}{2}$. Note that $s \geq 1$, $t \geq 0$, $z \geq 0$ and $s \geq z$. For even l (then $b = k[t(4s+4) + 2z] + 2s + 2$ and $r = t(4s+4) + 2z$) we get

$$
\begin{array}{rll}
w_{1,r} = & (BG)^{s-z}(BG)^z RG(RB)^s GR & [(BG)^s RG(RB)^s GR]^{t-1}(GR)^s BG(BG)^z \\
& [[(RB)^s GR(BG)^s RG]^t (BG)^z]^{k-1} & [(RB)^s GR(BG)^s RG]^t (BG)^z \\
& (RB)^{s-z}(RB)^z GR(BG)^s RG & [(RB)^s GR(BG)^s RG]^{t-1}(RB)^s GR(GR)^z \\
& [[(BG)^s RG(RB)^s GR]^t (GR)^z]^{k-1} & [(BG)^s RG(RB)^s GR]^t (GR)^z \\
w_{1,b-2}\rho^b = & (RB)^{s-z}(GR)^z GR(BG)^s RG & [(RB)^s GR(BG)^s RG]^{t-1}(RB)^s GR(GR)^z \\
& [[(BG)^s RG(RB)^s GR]^t (GR)^z]^{k-1} & [(BG)^s RG(RB)^s GR]^t (GR)^z \\
& (GR)^{s-z}(BG)^z BG(RB)^s GR & [(BG)^s RG(RB)^s GR]^{t-1}(BG)^s RG(BG)^z \\
& [[(RB)^s GR(BG)^s RG]^t (BG)^z]^{k-1} & [(RB)^s GR(BG)^s RG]^t (RB)^z \\
w_{1,b-2}\rho^m = & (RB)^{s-z}(RB)^z GR(BG)^s RG & [(RB)^s GR(BG)^s RG]^{t-1}(RB)^s GR(GR)^z \\
& [[(BG)^s RG(RB)^s GR]^t (GR)^z]^{k-1} & [(BG)^s RG(RB)^s GR]^t (GR)^z \\
& (BG)^{s-z}(BG)^z RG(RB)^s GR & [(BG)^s RG(RB)^s GR]^{t-1}(GR)^s BG(BG)^z \\
& [[(RB)^s GR(BG)^s RG]^t (BG)^z]^{k-1} & [(RB)^s GR(BG)^s RG]^t (BG)^z
\end{array}
$$

while for odd $l \geq 3$ (then $b = k[t(4s+4)+2s+2z+2]+2s+2$ and $r = t(4s+4)+2s+2z+2$)

$$
\begin{array}{rll}
w_{1,r} = & GR(BG)^{s-z}(BG)^z(RB)^s GR & [GR(BG)^s(RB)^s GR]^{t-1}(BG)^s RB(GR)^s BG(BG)^z \\
& [[(RB)^s GR(BG)^s RB]^t (GR)^s BG(BG)^z]^{k-1} & [(RB)^s GR(BG)^s RB]^t (GR)^s BG(BG)^z \\
& (RB)^{s-z}(RB)^z GR(BG)^s RB & [(RB)^s GR(BG)^s RB]^{t-1}(GR)^s BG(RB)^s GR(GR)^z \\
& [[(BG)^s RB(RB)^s GR]^t (BG)^s RB(GR)^z]^{k-1} & [(BG)^s RB(RB)^s GR]^t (BG)^s RB(GR)^z \\
w_{1,b-2}\rho^b = & (RB)^{s-z} GR(GR)^z(BG)^s RB & [(RB)^s GR(BG)^s RB]^{t-1}(RB)^s GR(BG)^s RB(GR)^z \\
& [[(BG)^s RB(RB)^s GR]^t (BG)^s RB(GR)^z]^{k-1} & [GR(BG)^s(RB)^s GR]^t (BG)^s RB(GR)^z \\
& (GR)^{s-z} BG(BG)^z(RB)^s GR & [(BG)^s RB(RB)^s GR]^{t-1}(BG)^s RB(GR)^s BG(BG)^z \\
& [[(RB)^s GR(BG)^s RB]^t (GR)^s BG(BG)^z]^{k-1} & [(RB)^s GR(BG)^s RB]^t (GR)^s BG(RB)^z \\
w_{1,b-2}\rho^m = & (RB)^{s-z}(RB)^z GR(BG)^s RB & [(RB)^s GR(BG)^s RB]^{t-1}(GR)^s BG(RB)^s GR(GR)^z \\
& [[(BG)^s RB(RB)^s GR]^t (BG)^s RB(GR)^z]^{k-1} & [(BG)^s RB(RB)^s GR]^t (BG)^s RB(GR)^z \\
& GR(BG)^{s-z}(BG)^z(RB)^s GR & [GR(BG)^s(RB)^s GR]^{t-1}(BG)^s RB(GR)^s BG(BG)^z \\
& [[(RB)^s GR(BG)^s RB]^t (GR)^s BG(BG)^z]^{k-1} & [(RB)^s GR(BG)^s RB]^t (GR)^s BG(BG)^z
\end{array}
$$

and for $l = 1$ (then $b = k[2s+2z+2]+2s+2$ and $r = 2s+2z+2$)

$$
\begin{array}{rll}
w_{1,r} = & (BG)^s RB(GR)^s(BG)^z RB & [(GR)^s(BG)^z RB]^{k-1}(GR)^s(BG)^z RB \\
& (GR)^s BG(RB)^s GR(GR)^z & [(BG)^s RB(GR)^z]^{k-1}(BG)^s RB(GR)^z \\
w_{1,r}\rho^b = & (RB)^{s-z}GR(GR)^z(BG)^s RB(GR)^z & [(BG)^s RB(GR)^z]^{k-1}(BG)^s RB(GR)^z \\
& (BG)^{s-z}(BG)^z RB(GR)^s(BG)^z RB & [(GR)^s(BG)^z RB]^{k-1}(GR)^s BG(RB)^z \\
w_{1,r}\rho^m = & (GR)^s BG(RB)^s GR(GR)^z & [(BG)^s RB(GR)^z]^{k-1}(BG)^s RB(GR)^z \\
& (BG)^s RB(GR)^s(BG)^z RB & [(GR)^s(BG)^z RB]^{k-1}(GR)^s(BG)^z RB.
\end{array}
$$

I.B. Let $s = \frac{p-2}{2}$, $t = \lfloor \frac{l-1}{2} \rfloor$ and $z = \frac{q-1}{2}$. Note that $s \geq 1$, $t \geq 0$, $z \geq 0$ and $s \geq z$. For even l (then $b = k[(t+1)(4s+4)+2z+1]+2s+2$ and $r = (t+1)(4s+4)+2z+1$) we get

$$
\begin{array}{rll}
w_{1,r} = & R(GR)^{s-z}(GR)^z B(GR)^s BG & [(RG)^s RB(GR)^s BG]^t RB(GB)^{s-1}GR(BR)^z B \\
& [[(GR)^s BG(RB)^s GR]^t (GR)^s BG(RB)^s GRB(RB)^z]^{k-1} & [(GR)^s BG(RB)^s GR]^t (GR)^s BG(RB)^s GRB(RB)^z \\
& (GR)^{s-z}(GR)^z BG(RB)^s GR & [(GR)^s BG(RB)^s GR]^t (BG)^{z+1}\omega(RB)^z G \\
& [[(RB)^s GR(GR)^s BG]^t (RB)^s GR(BG)^s RBG(BG)^z]^{k-1} & [(RB)^s GR(GR)^s BG]^t (RB)^s GR(BG)^s RBG(BG)^z \\
w_{1,r}\rho^b = & G\omega(RB)^z G(RB)^s GR & [(GR)^s BG(RB)^s GR]^t (BG)^s RB(GB)^z G \\
& [[(RB)^s GR(GR)^s BG]^t (RB)^s GR(BG)^s RBG(BG)^z]^{k-1} & [(RG)^s RB(GR)^s BG]^t (RG)^s RB(GR)^s BGR(BG)^z \\
& (BG)^{s-z}(RB)^z RB(GR)^s BG & [(RB)^s GR(GR)^s BG]^t (RB)^s GR(BR)^z B \\
& [[(GR)^s BG(RB)^s GR]^t (GR)^s BG(RB)^s GRB(RB)^z]^{k-1} & [(GR)^s BG(RB)^s GR]^t (GR)^s BG(RB)^s GRB(GB)^z \\
w_{1,r}\rho^m = & (GR)^{s-z}(GR)^z BG(RB)^s GR & [(GR)^s BG(RB)^s GR]^t (BG)^{z+1}\omega(RB)^z G \\
& [[(RB)^s GR(GR)^s BG]^t (RB)^s GR(BG)^s RBG(BG)^z]^{k-1} & [(RB)^s GR(GR)^s BG]^t (RB)^s GR(BG)^s RBG(BG)^z \\
& R(GR)^{s-z}(GR)^z B(GR)^s BG & [(RG)^s RB(GR)^s BG]^t RB(GB)^{s-1}GR(BR)^z B \\
& [[(GR)^s BG(RB)^s GR]^t (GR)^s BG(RB)^s GRB(RB)^z]^{k-1} & [(GR)^s BG(RB)^s GR]^t (GR)^s BG(RB)^s GRB(RB)^z
\end{array}
$$

while for odd $l \geq 3$ (then $b = k[t(4s+4)+2s+2z+3]+2s+2$ and $r = t(4s+4)+2s+2z+3$)

$$\begin{array}{rll}
w_{1,r} = & (RB)^{s-z}(RB)^z GR(BR)^s BG & [(RB)^s GR(BR)^s BG]^{t-1}(RB)^s GR(GR)^s GR(BG)^z B \\
& [[(GR)^s BG(RB)^s GR]^t (GR)^s BG(BG)^z R]^{k-1} & [(GR)^s BG(RB)^s GR]^t (GR)^s BG(BG)^z R \\
& (BR)^{s-z} B(RB)^z G(RB)^s GR & [(BR)^s BG(RB)^s GR]^{t-1}(BR)^s BG(RG)^s RB(GB)^z G \\
& [[(RB)^s GR(BR)^s BG]^t (RB)^s RB(GB)^z G]^{k-1} & [(RB)^s GR(BR)^s BG]^t (RB)^s RB(GB)^z G \\
w_{1,r}\rho^b = & (GR)^{s-z}(BG)^z BG(RB)^s GR & [(BR)^s BG(RB)^s GR]^{t-1}(BR)^s BG(RB)^s RB(GB)^z G \\
& [[(RB)^s GR(BR)^s BG]^t (RB)^s RB(GB)^z G]^{k-1} & [(RB)^s GR(BR)^s BG]^t (RB)^s GR(GR)^z G \\
& (RG)^{s-z} R(BG)^z B(GR)^s BG & [(RB)^s GR(GR)^s BG]^{t-1}(RB)^s GR(GR)^s BG(BG)^z R \\
& [[(GR)^s BG(RB)^s GR]^t (GR)^s BG(BG)^z R]^{k-1} & [(BR)^s BG(RB)^s GR]^t (BR)^s BG(RG)^z R \\
w_{1,r}\rho^m = & (BR)^{s-z} B(RB)^z G(RB)^s GR & [(BR)^s BG(RB)^s GR]^{t-1}(BR)^s BG(RG)^s RB(GB)^z G \\
& [[(RB)^s GR(BR)^s BG]^t (RB)^s RB(GB)^z G]^{k-1} & [(RB)^s GR(BR)^s BG]^t (RB)^s RB(GB)^z G \\
& (RB)^{s-z}(RB)^z GR(BR)^s BG & [(RB)^s GR(BR)^s BG]^{t-1}(RB)^s GR(GR)^s GR(BG)^z B \\
& [[(GR)^s BG(RB)^s GR]^t (GR)^s BG(BG)^z R]^{k-1} & [(GR)^s BG(RB)^s GR]^t (GR)^s BG(BG)^z R
\end{array}$$

and for $l = 1$ (then $b = k[2s+2z+3]+2s+2$ and $r = 2s+2z+3$)

$$\begin{array}{rl}
w_{1,r} = & (RB)^{s-z}(RB)^z GR(GR)^s GR(BG)^z B[(GR)^s BG(BG)^z R]^{k-1}(GR)^s BG(BG)^z R \\
& (BR)^{s-z} B(RB)^z G(RG)^s RB(GB)^z G[(RB)^s RB(GB)^z G]^{k-1}(RB)^s RB(GB)^z G \\
w_{1,r}\rho^b = & (GR)^{s-z}(BG)^z BG(RB)^s RB(GB)^z G[(RB)^s RB(GB)^z G]^{k-1}(RB)^s GR(GR)^z G \\
& (RG)^{s-z} R(BG)^z B(GR)^s BG(BG)^z R[(GR)^s BG(BG)^z R]^{k-1}(BR)^s BG(RG)^z R \\
w_{1,r}\rho^m = & (BR)^{s-z} B(RB)^z G(RG)^s RB(GB)^z G[(RB)^s RB(GB)^z G]^{k-1}(RB)^s RB(GB)^z G \\
& (RB)^{s-z}(RB)^z GR(GR)^s GR(BG)^z B[(GR)^s BG(BG)^z R]^{k-1}(GR)^s BG(BG)^z R.
\end{array}$$

I.C. Let $s = \frac{p-1}{2}$, $t = \frac{l-1}{2}$ and $z = \frac{q}{2}$. Note that $s \geq 1$, $t \geq 0$, $z \geq 0$ and $s \geq z$. For $l \geq 3$ (then $b = k[t(4s+2)+2s+2z+1]+2s+1$ and $r = t(4s+2)+2s+2z+1$) we get

$$\begin{array}{rll}
w_{1,r} = & (RB)^s R(BG)^s B & [(RB)^s R(BG)^s B]^{t-1}(RB)^s R(GB)^s G(BG)^z \\
& [[(BG)^s B(RB)^s R]^t (GR)^s G(BG)^z]^{k-1} & [(BG)^s B(RB)^s R]^t (GR)^s G(BG)^z \\
& (BG)^s B(RB)^s R & [(BG)^s B(RB)^s R]^{t-1}(BR)^s G(BG)^s B(GB)^z \\
& [[(RB)^s R(BG)^s B]^t (RG)^s B(GB)^z]^{k-1} & [(RB)^s R(BG)^s B]^t (RG)^s B(GB)^z \\
w_{1,r}\rho^b = & (BG)^s B(RB)^s R & [(BG)^s B(RB)^s R]^{t-1}(BG)^s B(RG)^s B(GB)^z \\
& [[(RB)^s R(BG)^s B]^t (RG)^s B(GB)^z]^{k-1} & [(RB)^s R(BG)^s B]^t (RB)^s R(GB)^z \\
& (GB)^s G(BG)^s B & [(RB)^s R(BG)^s B]^{t-1}(RB)^s R(GR)^s G(BG)^z \\
& [[(BG)^s B(RB)^s R]^t (GR)^s G(BG)^z]^{k-1} & [(BG)^s B(RB)^s R]^t (BR)^s G(BG)^z \\
w_{1,r}\rho^m = & (BG)^s B(RB)^s R & [(BG)^s B(RB)^s R]^{t-1}(BR)^s G(BG)^s B(GB)^z \\
& [[(RB)^s R(BG)^s B]^t (RG)^s B(GB)^z]^{k-1} & [(RB)^s R(BG)^s B]^t (RG)^s B(GB)^z \\
& (RB)^s R(BG)^s B & [(RB)^s R(BG)^s B]^{t-1}(RB)^s R(GB)^s G(BG)^z \\
& [[(BG)^s B(RB)^s R]^t (GR)^s G(BG)^z]^{k-1} & [(BG)^s B(RB)^s R]^t (GR)^s G(BG)^z
\end{array}$$

and for $l = 1$ (then $b = k[2s+2z+1]+2s+1$ and $r = 2s+2z+1$)

$$\begin{array}{rl}
w_{1,r} = & (RB)^s R(GB)^s G(BG)^{z-1} BR[(GR)^s G(BG)^{z-1} BR]^{k-1}(GR)^s G(BG)^{z-1} BR \\
& (BR)^s G(BG)^s B(GB)^{z-1} GB[(RG)^s B(GB)^{z-1} GB]^{k-1}(RG)^s B(GB)^{z-1} GB \\
w_{1,r}\rho^b = & (BG)^s B(RG)^s B(GB)^{z-1} GB[(RG)^s B(GB)^{z-1} GB]^{k-1}(RB)^s R(GB)^{z-1} GB \\
& (GB)^s R(GR)^s G(BG)^{z-1} BR[(GR)^s G(BG)^{z-1} BR]^{k-1}(BR)^s G(BG)^{z-1} BG \\
w_{1,r}\rho^m = & (BR)^s G(BG)^s B(GB)^{z-1} GB[(RG)^s B(GB)^{z-1} GB]^{k-1}(RG)^s B(GB)^{z-1} GB \\
& (RB)^s R(GB)^s G(BG)^{z-1} BR[(GR)^s G(BG)^{z-1} BR]^{k-1}(GR)^s G(BG)^{z-1} BR.
\end{array}$$

I.D. Let $s=\frac{p-1}{2}$, $t=\frac{l-2}{2}$ and $z=\frac{q-1}{2}$. Note that $s\geq 1$, $t\geq 0$, $z\geq 0$, $s\geq z$ and moreover $\frac{lp-p-1}{2}=2st+t+s$. For $l\geq 4$ (then $b=k[4ts+2t+4s+2z+3]+2s+1$ and $r=4ts+2t+4s+2z+3$) we get

$$\begin{aligned}
w_{1,r}=&(BG)^sB(GB)^sG && (BG)^{2st+t-s}(RG)^s(RB)^{s+1}(RB)^z\\
&[(GB)^{2st+t+s}(GR)^{s+1}(GR)^zB]^{k-1} && (GB)^{2st+t+s}GR(GR)^s(GR)^zB\\
&(GB)^sG(BG)^sB && (GB)^{2st+t-s}(GR)^s(BR)^sBR(BR)^z\\
&[(BG)^{2st+t+s}(RG)^{s+1}(RG)^zR]^{k-1} && (BG)^{2st+t+s}RG(RG)^s(RG)^zR\\
w_{1,r}\rho^b=&(RB)^sR(BG)^sB && (GB)^{2st+t-s}(GB)^{s-1}GR(GR)^{s+1}(GR)^z\\
&[(BG)^{2st+t+s}(RG)^{s+1}(RG)^zR]^{k-1} && (BG)^{2st+t+s}BG(RG)^s(RB)^zR\\
&(BR)^sB(GB)^sG && (BG)^{2st+t-s}(BG)^s(RG)^s(RG)^zRB\\
&[(GB)^{2st+t+s}(GR)^{s+1}(GR)^zB]^{k-1} && (GB)^{2st+t+s}GB(GR)^s(BR)^zB\\
w_{1,r}\rho^m=&(GB)^sG(BG)^sB && (GB)^{2st+t-s}(GR)^s(BR)^{s+1}(BR)^z\\
&[(BG)^{2st+t+s}(RG)^{s+1}(RG)^zR]^{k-1} && (BG)^{2st+t+s}RG(RG)^s(RG)^zR\\
&(BG)^sB(GB)^sG && (BG)^{2st+t-s}(RG)^s(RB)^sRB(RB)^z\\
&[(GB)^{2st+t+s}(GR)^{s+1}(GR)^zB]^{k-1} && (GB)^{2st+t+s}GR(GR)^s(GR)^zB
\end{aligned}$$

and for $l=2$ (then $b=k[4s+2z+3]+2s+1$ and $r=4s+2z+3$)

$$\begin{aligned}
w_{1,r}=&(BG)^sB(GR)^sG(RB)^{s+1}(RB)^z\ [(GB)^s(GR)^{s+1}(GR)^zB]^{k-1}(GB)^sGR(GR)^s(GR)^zB\\
&(GB)^sGB(GR)^s(BR)^sBR(BR)^z[(BG)^s(RG)^{s+1}(RG)^zR]^{k-1}(BG)^sRG(RG)^s(RG)^zR\\
w_{1,r}\rho^b=&(RB)^sR(BG)^sR(GR)^{s+1}(GR)^z\ [(BG)^s(RG)^{s+1}(RG)^zR]^{k-1}(BG)^sBG(RG)^s(RB)^zR\\
&(BR)^s(BG)^sBG(RG)^s(RG)^zRB[(GB)^s(GR)^{s+1}(GR)^zB]^{k-1}(GB)^sGB(GR)^s(BR)^zB\\
w_{1,r}\rho^m=&(GB)^sGB(GR)^s(BR)^{s+1}(BR)^z\ [(BG)^s(RG)^{s+1}(RG)^zR]^{k-1}(BG)^sRG(RG)^s(RG)^zR\\
&(BG)^sBG(RG)^s(RB)^sRB(RB)^z[(GB)^s(GR)^{s+1}(GR)^zB]^{k-1}(GB)^sGR(GR)^s(GR)^zB.
\end{aligned}$$

II.A. For $b=3i+4$ (then $r=3i+2$), where $i\geq 0$, we get

$$\begin{aligned}
w_{1,r}=&(BGR)^iBGBG(RBG)^iRB(GRB)^iRBGR(BGR)^iGR\\
w_{1,r}\rho^b=&(GRB)^iGRGR(BGR)^iBG(BGR)^iBGRB(GRB)^iRB\\
w_{1,r}\rho^m=&(GRB)^iRBGR(BGR)^iGR(BGR)^iBGBG(RBG)^iRB
\end{aligned}$$

for $b=3i+6$, where $i\geq 1$ (then $r=3i+4$), we get

$$\begin{aligned}
w_{1,r}=&BG(BGR)^iBGBGRB(RBG)^iRB\,RBGRB(GRB)^{i-1}RBGR\,GR(BGR)^iGR\\
w_{1,r}\rho^b=&GR(GRB)^iGRGRBG(BGR)^iBGBGRBR(BGR)^{i-1}BGRB\,RB(GRB)^iRB\\
w_{1,r}\rho^m=&RB(GRB)^iRBGR\,GR(BGR)^iGRBGBGR(BGR)^{i-1}BGBGRB(RBG)^iRB
\end{aligned}$$

while for $b=3i+11$ (then $r=3i+9$), where $i\geq 1$, we get

$$\begin{aligned}
w_{1,r}=&GRBG(BGR)^iBGBGRBRBGRB(GRB)^{i-1}GBGRBRBG\\
&RGRB(GRB)^iRBGRGRB\,G(BGR)^iBGRGRBGB\\
w_{1,r}\rho^b=&RBGB(GRB)^iGRGRBGBGRBG(BGR)^{i-1}BGRBGBGR\\
&BRBG(RBG)^iBGRBRBGR(GRB)^iGRBRBGRG\\
w_{1,r}\rho^m=&RGRB(GRB)^iRBGRGRB\,GBGR(BGR)^{i-1}BGRGRBGB\\
&GRBG(BGR)^iBGBGRBRB(GRB)^iGBGRBRBG.
\end{aligned}$$

II.B. Let $z=\frac{r}{2}$. Then $b=2kz+2$ and $r=2z$. Note that $z\geq 1$. We get

$$\begin{aligned}
w_{1,r}=&(GR)^{kz-z}(BG)^zBG(BG)^z(RB)^{kz-2z}(RB)^z(RB)^zGR\,(GR)^z\\
w_{1,r}\rho^b=&(RB)^{kz-z}(GR)^zGR(GR)^z(GR)^{kz-2z}(BG)^z(BG)^zBG(RB)^z\\
w_{1,r}\rho^m=&(RB)^{kz-z}(RB)^zGR\,(GR)^z(GR)^{kz-2z}(GR)^z(BG)^zBG(BG)^z.
\end{aligned}$$

II.C. For $r \geq 5$ and $k \geq 2$ (where $b = k(2l+3)+2$ and $r = 2l+3$) we get

$$\begin{aligned} w_{1,r} =& BGR(GR)^l[BGR(BG)^l]^{k-2}BGRBG(BG)^lRBG(BG)^l \\ & RBG(RB)^l[GRB(RB)^l]^{k-2}GRB(RB)^lRB\ GRB(GR)^l \\ w_{1,r}\rho^b =& GRB(RB)^l[GRB(RB)^l]^{k-2}RBGRB(GR)^l\ BGR(GR)^l \\ & BGR(BG)^l[BGR(BG)^l]^{k-2}BGR(BG)^lBGRBG(RB)^l \\ w_{1,r}\rho^m =& RBG(RB)^l[GRB(RB)^l]^{k-2}GRBRB(RB)^l\ GRB(GR)^l \\ & BGR(GR)^l[BGR(BG)^l]^{k-2}BGR(BG)^lBGRBG(BG)^l \end{aligned}$$

while for $r = 3$ and $k \geq 4$ (where $b = 3k+2$) we get

$$\begin{aligned} w_{1,r} =& BGR[GRB]^{k-4}GRBGRBRBGRBGRBRBG[RBG]^{k-4}RBGBGRBGRBGBGR \\ w_{1,r}\rho^b =& RBG[RBG]^{k-4}BGRBGRBGBGRBGRGRB[GRB]^{k-4}GRBRBGRBGRBRBG \\ w_{1,r}\rho^m =& RBG[RBG]^{k-4}RBGBGRBGRBGBGRBGR[GRB]^{k-4}GRBGRBRBGRBGRB. \end{aligned}$$

III. Let $z = \frac{r-1}{2}$. Then $b = k(2z+1)+1$ and $r = 2z+1$. Note that $z \geq 1$. We get

$$\begin{aligned} w_{1,r} =& [(GR)^zB]^{k-1}(GR)^zBG(RB)^zG[(RB)^zG]^{k-1}(RB)^zGR(GR)^zB \\ w_{1,r}\rho^b =& [(RB)^zG]^{k-1}(RG)^zRB(GR)^zB[(GR)^zB]^{k-1}G(RB)^zG(RB)^zG \\ w_{1,r}\rho^m =& [(RB)^zG]^{k-1}(RB)^zGR(GR)^zB[(GR)^zB]^{k-1}(GR)^zBG(RB)^zG. \end{aligned}$$

IV.A. Let $z = \frac{r}{2}$. Then $b = 2kz$ and $r = 2z$. Note that $z \geq 1$. We get

$$\begin{aligned} w_{1,r} =& (BG)^{zk-2z}(BG)^z(BG)^z(RB)^z(RB)^{zk-2z}(RB)^z(GR)^z(GR)^z \\ w_{1,r}\rho^b =& (RB)^{zk-2z}(GR)^z(GR)^z(BG)^z(BG)^{zk-2z}(BG)^z(RB)^z(RB)^z \\ w_{1,r}\rho^m =& (RB)^{zk-2z}(RB)^z(GR)^z(GR)^z(BG)^{zk-2z}(BG)^z(BG)^z(RB)^z. \end{aligned}$$

IV.B. Let $z = \frac{r-1}{2}$. Then $b = k(2z+1)$ and $r = 2z+1$. Note that $z \geq 1$. We get

$$\begin{aligned} w_{1,r} =& [(GR)^zB]^{k-2}(GR)^zB(GR)^zB(RB)^zG[(RB)^zG]^{k-2}(RB)^zG(BG)^zR(BG)^zR \\ w_{1,r}\rho^b =& [(RB)^zG]^{k-2}(BG)^zR(BG)^zR(GR)^zB[(GR)^zB]^{k-2}(GR)^zB(RB)^zG(RB)^zG \\ w_{1,r}\rho^m =& [(RB)^zG]^{k-2}(RB)^zG(BG)^zR(BG)^zR[(GR)^zB]^{k-2}(GR)^zB(GR)^zB(RB)^zG. \quad \square \end{aligned}$$

A series of the above lemmas yield the following theorem which characterizes 3-colorable circulants with $a = 1$.

Theorem 10. *Let n be even. A circulant $G = C(n; 1, b, \frac{n}{2})$ is 3-colorable if $b \notin \{2, \frac{n}{4}, \frac{n}{2}-1\}$ and G is not one of the following 9 circulants:*
$G \neq C(16; 1, 3, 8)$ *and*
$G \neq C(16; 1, 5, 8)$ *and*
$G \neq C(20; 1, 6, 10)$ *and*
$G \neq C(22; 1, 8, 11)$ *and*
$G \neq C(24; 1, 5, 12)$ *and*
$G \neq C(28; 1, 5, 14)$ *and*
$G \neq C(28; 1, 8, 14)$ *and*
$G \neq C(28; 1, 11, 14)$ *and*
$G \neq C(40; 1, 11, 20)$. □

The next four lemmas determine the chromatic number of circulants with $a = 1$ which are not 3-colorable.

Lemma 11. *Let $n \geq 8$ be even. The circulant $G = C(n; 1, 2, \frac{n}{2})$ has $\chi(G) = 4$.*

Proof. Assume that G is 3-colorable. W.l.o.g. we may color vertices 0, 1 and 2 by colors B, G and R, respectively. Next we color consecutive vertices: $3, 4, \ldots, n-1$; each of them is uniquely colored. To end up with such a coloring the condition $3 \mid n$ has to be satisfied. Hence $3 \mid \frac{n}{2}$, which means that vertices 0 and $\frac{n}{2}$ got the same color, a contradiction.

G has the following 4-coloring sequence w_n over the alphabet $\{B, G, R, Y\}$. If $n \equiv 0 \pmod 8$ then $w_n = (BGRY)^{\frac{n}{8}}(GBYR)^{\frac{n}{8}}$; if $n \equiv 2 \pmod 8$ then $w_n = (BGRY)^{\frac{n-2}{8}}B(RYBG)^{\frac{n-2}{8}}R$; if $n \equiv 4 \pmod 8$ then $w_n = (BGRY)^{\frac{n}{4}}$; moreover, if $n \equiv 6 \pmod 8$ then $w_n = (BGRY)^{\frac{n-6}{8}}GBR(YBGR)^{\frac{n-6}{8}}BGY$. $\square$

Lemma 12. *Let $n \geq 8$ and $4 \mid n$. The circulant $G = C(n; 1, \frac{n}{4}, \frac{n}{2})$ has $\chi(G) = 4$.*

Proof. Notice that the vertices $0, \frac{n}{4}, \frac{n}{2}$ and $\frac{3n}{4}$ induce in G a clique of order 4. Hence $\chi(G) \geq 4$.

A 4-coloring sequence of G over the alphabet $\{B, G, R, Y\}$ is $w_n = (BG)^{\frac{n}{8}}(RY)^{\frac{n}{8}}(GB)^{\frac{n}{8}}(YR)^{\frac{n}{8}}$ if $8 \mid n$ and $w_n = (BG)^{\frac{n-4}{8}}B(RY)^{\frac{n-4}{8}}R(GB)^{\frac{n-4}{8}}G(YR)^{\frac{n-4}{8}}Y$ otherwise. $\square$

Lemma 13. *Let $n \geq 8$ be even. The circulant $G = C(n; 1, n-1, \frac{n}{2})$ has $\chi(G) = 4$ if $4 \mid n$ and $\chi(G) = 5$ otherwise.*

Proof. Notice that the vertices $0, 1, \frac{n}{2}$ and $\frac{n}{2} + 1$ induce in G a clique of order 4. Hence $\chi(G) \geq 4$. Assume that $n \equiv 2 \pmod 4$ and G is 4-colorable. W.l.o.g. we may color vertices $0, 1, \frac{n}{2}$ and $\frac{n}{2} + 1$ by colors B, G, R and Y, respectively. Then to color two neighboring vertices 2 and $\frac{n}{2} + 2$ we may use only colors B and R; that means either 2 is colored by B and $\frac{n}{2} + 2$ by R or vice versa. To color the next pair of vertices (i.e. 3 and $\frac{n}{2} + 3$) the available colors are G and Y. We continue in the same manner. Finally, the pair of vertices $\frac{n}{2} - 2$ and $n - 2$ get colors G and Y. Hence the vertex $\frac{n}{2} - 1$ cannot be properly colored.

If $n \equiv 0 \pmod 4$ then a 4-coloring sequence of G over the alphabet $\{B, G, R, Y\}$ is $w_n = (BG)^{\frac{n}{4}}(RY)^{\frac{n}{4}}$. In the case $n \equiv 2 \pmod 4$ a 5-coloring sequence of G over $\{B, G, R, Y, W\}$ is $w_n = (BG)^{\frac{n-2}{4}}W(RY)^{\frac{n-6}{4}}WRY$. $\square$

Lemma 14. *The circulants: $C(16; 1, 3, 8)$, $C(16; 1, 5, 8)$, $C(20; 1, 6, 10)$, $C(22; 1, 8, 11)$, $C(24; 1, 5, 12)$, $C(28; 1, 5, 14)$, $C(28; 1, 8, 14)$, $C(28; 1, 11, 14)$ and $C(40; 1, 11, 20)$ have chromatic number 4.*

Proof. Let G denote one of the listed circulants; these are the exceptional circulants of Theorem 10. By the proofs of Lemmas 8 and 9 we get $\chi(G) > 3$. Let $u_{n,b}$ denote a coloring sequence of G over the alphabet $\{B, G, R, Y\}$. If b is odd then $4 \mid n$ and we take $u_{n,b} = (BG)^{n/4}(RY)^{n/4}$. Moreover we define $u_{20,6} = (BGRY)^5$, $u_{22,8} = (BGRY)^2GRY(RYBG)^2YBG$ and $u_{28,8} = (BG)^4(RY)^3(GB)^4(YR)^3$. $\square$

4. Case $a \geq 2$

By Claim 2 we may assume $\gcd(a, b) \leq 2$. In the case $\gcd(a, b) = 2$ (then $n \equiv 2 \pmod 4$ by Claim 2) the following proposition determines the chromatic number.

Proposition 15. *Let $n \equiv 2 \pmod 4$, $n \geq 10$ and $\gcd(a, b) = 2$.*
Then $\chi(G) = \chi(C(n/2; a/2, b/2))$. In particular,
$\chi(G) = 4$ if $n \equiv \pm 1 \bmod 3$, $n \neq 10$, and ($b \equiv \pm 2a \pmod n$ or $a \equiv \pm 2b \pmod n$),
$\chi(G) = 4$ if $n = 26$, and ($b \equiv \pm 5a \pmod{26}$ or $a \equiv \pm 5b \pmod{26}$),
$\chi(G) = 5$ if $n = 10$,
and $\chi(G) = 3$ otherwise.

Proof. Here $n/2$ is odd. Take the subgroup of index 2 of $\mathbb{Z}_n$ consisting of even numbers. Since both a, b are even, the respective induced subgraph of G is $C(n/2; a/2, b/2)$. Hence $\chi(G) \geq \chi(C(n/2; a/2, b/2))$. On the other hand, G consists of two copies, H_0,H_1, of $C(n/2; a/2, b/2)$ joined by edges of the form $(i, i + n/2)$. Let ϕ be a coloring of H_0 employing minimum number of colors $m = \chi(G(n/2; a/2, b/2))$. Clearly, G admits a cyclic automorphism α taking $x \mapsto x + a$. Hence the equation $\psi(x + n/2 + a) = \phi(x)$ defines a minimum coloring on vertices of H_1. We claim that $\phi \cup \psi$ is a proper coloring of G. Indeed, let x and $x + n/2$ be two vertices. Let c be the color of x. Then by definition $x + n/2 + a$ is colored by c as well. Since the coloring is proper on H_1, the color of $x + n/2$ is different from c. Thus the coloring $\phi \cup \psi$ is proper proving $\chi(G) \leq \chi(C(n/2; a/2, b/2))$. The remaining part of the statement follows from Theorem 3 in [2] classifying the chromatic number of 4-valent circulants. Since $n/2$ is odd, G cannot be bipartite. $\square$

In what follows we consider the case when $\gcd(a, b) = 1$. If $b \equiv \pm 2a \pmod n$ or $a \equiv \pm 2b \pmod n$ then either $a = 1$ or $b = 1$. Moreover, if $a + b = n/2$ then clearly $\gcd(a, n) = 1$ or $\gcd(b, n) = 1$.

Suppose that $\gcd(a, n) = 1$ or $\gcd(b, n) = 1$. The chromatic number of G is determined by Theorem 10, Lemmas 11–14 and Claim 3.

Therefore in what follows we assume $\gcd(a, n) > 1$ and $\gcd(b, n) > 1$.

Theorem 16. *Let n be even. If $\gcd(a, b) = 1$, $\gcd(a, n) > 1$, $\gcd(b, n) > 1$ and $a, b \neq \frac{n}{4}$, then a circulant $G = C(n; a, b, \frac{n}{2})$ is 3-colorable.*

Proof. Since $\gcd(a, b) = 1$, at least one of the integers a, b is odd. Hence G is connected. W.l.o.g. we may assume a is odd.

Let $n = 2m$, $g = \gcd(a, n)$ and $\lambda = \frac{n}{g}$. Notice that g (≥ 3) is odd and λ is even, moreover $\lambda \geq 6$. Denote by $H = \langle a \rangle$ the cyclic subgroup of $\mathbb{Z}_n$ generated by a. Clearly, λ is the order of H and $\frac{\lambda}{2} a \equiv m \pmod n$. The elements of $\mathbb{Z}_n$ can be decomposed into g cosets of the form $H + jb$, $j = 0, 1, \ldots, g - 1$. Let the elements in a coset $H + jb$ be ordered as follows: $jb, jb + a, jb + 2a, \ldots, jb + (\lambda - 1)a$. Clearly, gb is the element of the coset $H + 0b$. Thus $gb \equiv xa \pmod n$ for some x such that $0 \leq x \leq \lambda - 1$. Since one of x, $\lambda - x$ is not greater than $\lambda/2$ and $(n - x)a$ is the element of $H + 0b$, w.l.o.g. we may assume that $0 \leq x \leq \lambda/2$. For each element f of $\mathbb{Z}_n$ there exists a unique couple $(i, j) \in \mathbb{Z}_\lambda \times \mathbb{Z}_g$ such that $f = ia + jb$, where j and i uniquely determine a coset and a position of f in $H + jb$, respectively.

A 3-coloring of G can be defined by specifying g coloring cyclic sequences ψ_j, $j = 0, 1, \ldots, g - 1$, each of length λ, over the alphabet $\{B = 0, G = 1, R = 2\}$. Let $c_{i,j}$ denote a color of the element $ia + jb \in \mathbb{Z}_n$. We have to guarantee that:

(1) $c_{i,j} \neq c_{i+1,j}$ for each $i \in \mathbb{Z}_\lambda$ and each $j \in \mathbb{Z}_g$,
(2) $c_{i,j} \neq c_{i+\lambda/2,j}$ for each $i \in \mathbb{Z}_\lambda$ and each $j \in \mathbb{Z}_g$,
(3) $c_{i,j} \neq c_{i,j+1}$ for each $i \in \mathbb{Z}_\lambda$ and each $j \in \mathbb{Z}_g - \{g - 1\}$,
(4) $c_{i,g-1} \neq c_{i+x,0}$ for each $i \in \mathbb{Z}_\lambda$.

Notice that elements of the coset $H + jb$ induce in G a 3-valent circulant isomorphic to $\bar{G} = C(\lambda; 1, \frac{\lambda}{2})$, for each $j \in \mathbb{Z}_g$. Since $\lambda \geq 6$, by Theorem 3 in [2], $\chi(\bar{G}) = 3$ if $\lambda \equiv 0 \pmod 4$ and $\chi(\bar{G}) = 2$ otherwise. In particular, $\gamma = (BG)^{\lambda/4} R (BG)^{\lambda/4-1} R$ and $\gamma = (BG)^{\lambda/2}$ are examples of proper colorings for $\lambda \equiv 0 \pmod 4$ and $\lambda \equiv 2 \pmod 4$, respectively.

Given a coloring cyclic sequence ψ, let $\psi\sigma^k$ denote the cyclic shift of ψ by k terms to the left. Consider separately the following cases.

I. $x = 0$. For even $j = 0, 2, \ldots, g - 3$ we define $\psi_0 = \gamma$. For odd $j = 1, 3, \ldots, g - 2$ as well as $j = g - 1$ we put $c_{i,j} = c_{i,j-1} + 1 \pmod 3$, where $i \in \mathbb{Z}_\lambda$. Conditions (1) and (2) are satisfied because γ is a proper coloring cyclic sequence of $\bar{G}$. Clearly, (3) and (4) are also guaranteed.

II. $x = 1, 3$. For even $j = 0, 2, \ldots, g - 3$ we define $\psi_j = \gamma$ while for odd $j = 1, 3, \ldots, g - 2$ we put $\psi_j = \gamma\sigma^1$. Moreover, we take $\psi_{g-1} = \gamma\sigma^2$. Since γ is a proper coloring cyclic sequence of $\bar{G}$, conditions (1) and (2) are immediately satisfied. Assume that $c_{i,j}$ corresponds to a kth term of γ. Then, for $j < g - 1$, $c_{i,j+1}$ is either a color of $(k + 1)$th term (if j is even or $j = g - 2$) or $(k - 1)$th term (if $j < g - 2$ is odd) of γ; thus (3) is guaranteed. Similarly, if $c_{i,g-1}$ corresponds to lth term of γ then $c_{i+x,0}$ is a color of $(l + 1)$th term (if $x = 3$) or $(l - 1)$th term (if $x = 1$) of γ, and therefore (4) is satisfied.

III. $x = 2$. For $\lambda \geq 12$ and $\lambda \neq 16$, by Theorem 10, a 5-valent circulant $\ddot{G} = C(\lambda; 1, 4, \frac{\lambda}{2})$ is 3-colorable. Moreover, for $\lambda = 8$, a 3-valent circulant $\ddot{G} = \bar{G} = C(\lambda; 1, \frac{\lambda}{2})$ is also 3-colorable. Let ς be a 3-coloring cyclic sequence of $\ddot{G}$. We take $\psi_j = \varsigma$ for even $j = 0, 2, \ldots, g - 3$, $\psi_j = \varsigma\sigma^4$ for odd $j = 1, 3, \ldots, g - 2$ and moreover $\psi_{g-1} = \varsigma\sigma^3$. Since ς is a proper coloring of $\ddot{G}$, conditions (1) and (2) are clearly satisfied. Let $c_{i,j}$ correspond to a kth term of ς. Then, for $j < g - 2$, $c_{i,j+1}$ is either a color of $(k + 4)$th term (if j is even) or $(k - 4)$th term (if j is odd) of ς. If $c_{i,g-2}$ corresponds to pth term of ς then $c_{i,g-1}$ is a color of $(p - 1)$th term of ς. Thus (3) is guaranteed. Moreover, if $c_{i,g-1}$ corresponds to lth term of ς then $c_{i+x,0}$ is a color of $(l - 1)$th term of ς, and therefore (4) is guaranteed.

In the remaining cases (for $\lambda = 6, 10, 16$) we define coloring cyclic sequences ω_λ: $\omega_6 = (BG)^3$, $\omega_{10} = (BG)^5$ and $\omega_{16} = (GB)^4 R (GB)^3 R$. Notice that these sequences are proper colorings of corresponding 3-valent circulants $\bar{G} = C(\lambda; 1, \frac{\lambda}{2})$ with the additional property that for any pair of terms c_i and c_{i+2} of ω_λ, we get $c_i + 2 \not\equiv c_{i+2} \pmod 3$.

For even $j = 0, 2, \ldots, g-3$ we take $\psi_j = \omega_\lambda$. For odd $j = 1, 3, \ldots, g-2$ as well as $j = g-1$ we put $c_{i,j} = c_{i,j-1} + 1 \pmod 3$, where $i \in \mathbb{Z}_\lambda$. Conditions (1) and (2) are satisfied because ω_λ is a proper coloring of $\bar{G}$. It is clear that (3) and (4) are also guaranteed.

IV. $4 \le x \le \lambda/2$. Let $\tilde{G}$ denote a 5-valent circulant $\tilde{G} = C(\lambda; 1, x, \frac{\lambda}{2})$. Assume $\tilde{G}$ is 3-colorable, by Theorem 10, and φ is a proper coloring cyclic sequence. We put $\psi_j = \varphi$ for even $j = 0, 2, \ldots, g-1$ and $\psi_j = \varphi\sigma^x$ for odd $j = 1, 3, \ldots, g-2$. Since φ is a proper coloring of $\tilde{G}$, conditions (1) and (2) are clearly satisfied. If $c_{i,j}$ corresponds to a kth term of φ then, for $j < g-1$, $c_{i,j+1}$ is either a color of $(k+x)$th term (if j is even) or $(k-x)$th term (if j is odd) of φ; thus (3) is guaranteed. Similarly, if $c_{i,g-1}$ corresponds to lth term of φ then $c_{i+x,0}$ is a color of $(l+x)$th term of φ, whence (4) is satisfied.

It remains to consider cases when $\tilde{G}$ is not 3-colorable; in almost all of those cases we define a circulant $\hat{G} = C(\lambda; 1, y, \frac{\lambda}{2})$ such that either $y = x+2$ or $y = x-2$. Assume that $x = \lambda/4$. Since $x \ge 4$ we immediately get $x + 2 < \frac{\lambda}{2} - 1$. Notice that if $x+2 = 6$, $x+2 = 8$ or $x+2 = 11$, then $\lambda = 16 \ne 20$, $\lambda = 24 \ne 22, 28$ or $\lambda = 36 \ne 28, 40$, respectively. Therefore we put $y = x+2$. Assume that $x = \frac{\lambda}{2} - 1$. For $x > 5$ we get $x - 2 > \lambda/4$. Notice that if $x - 2 = 6$ then $\lambda = 18 \ne 20$. If $x \ne 4, 5, 7, 10, 13$ we take $y = x-2$. Similarly, for the following pairs (x, λ): $(5, 24)$, $(5, 28)$, $(6, 20)$, $(8, 22)$, $(8, 28)$, $(11, 28)$ and $(11, 40)$ we put $y = x-2$. By Theorem 10, circulants $\hat{G}$ for y's specified above are 3-colorable. Let ϕ be a coloring cyclic sequence of $\hat{G}$. We take $\psi_j = \phi$ for even $j = 0, 2, \ldots, g-3$, $\psi_j = \phi\sigma^y$ for odd $j = 1, 3, \ldots, g-2$ and moreover $\psi_{g-1} = \phi\sigma^{y-1}$ if $y = x+2$ and $\psi_{g-1} = \phi\sigma^{y+1}$ otherwise. Since ϕ is a proper coloring of $\hat{G}$, conditions (1) and (2) are clearly satisfied. Let $c_{i,j}$ correspond to a kth term of ϕ. Then, for $j < g-2$, $c_{i,j+1}$ is either a color of $(k+y)$th term (if j is even) or $(k-y)$th term (if j is odd) of ϕ. If $c_{i,g-2}$ corresponds to pth term of ϕ then $c_{i,g-1}$ is a color of $(p+1)$th term (if $y = x-2$) or $(p-1)$th term (if $y = x+2$) of ϕ. Thus (3) is guaranteed. Similarly, if $c_{i,g-1}$ corresponds to lth term of ϕ then $c_{i+x,0}$ is a color of $(l+1)$th term (if $y = x-2$) or $(l-1)$th term (if $y = x+2$) of ϕ, and therefore (4) is satisfied.

Similarly as in III, in the remaining cases, i.e. for pairs (x, λ): $(4, 10)$, $(5, 12)$, $(5, 16)$, $(7, 16)$, $(10, 22)$ and $(13, 28)$, we define corresponding coloring cyclic sequences $\omega_{x,\lambda}$: $\omega_{4,10} = (BG)^5$, $\omega_{5,12} = (BG)^3R(BG)^2R$, $\omega_{5,16} = (BG)^3R(BG)^3RBR$, $\omega_{7,16} = (BG)^4R(BG)^3R$, $\omega_{10,22} = (BG)^{11}$ and $\omega_{13,28} = (BG)^7R(BG)^6R$, respectively. Since these sequences are proper colorings of $\bar{G} = C(\lambda; 1, \frac{\lambda}{2}\}$, conditions (1) and (2) are satisfied. The additional property, that for any pair of terms c_i and c_{i+x} of $\omega_{x,\lambda}$ we have $c_i + 2 \not\equiv c_{i+x} \pmod 3$, guarantees (3) and (4). Namely, for even $j = 0, 2, \ldots, g-3$ we take $\psi_j = \omega_{x,\lambda}$, while for odd $j = 1, 3, \ldots, g-2$ as well as $j = g-1$ we put $c_{i,j} = c_{i,j-1} + 1 \pmod 3$, where $i \in \mathbb{Z}_\lambda$.

V. $x = \lambda/2$. We define $\psi_j = \gamma$ for even $j = 0, 2, \ldots, g-3$, $\psi_j = \gamma\sigma^{\frac{\lambda}{2}}$ for odd $j = 1, 3, \ldots, g-2$, while $\psi_{g-1} = \gamma\sigma^{\frac{\lambda}{2}-1}$, where γ is a proper coloring cyclic sequence of $\bar{G}$. Thus conditions (1) and (2) are immediately satisfied. Assume that $c_{i,j}$ corresponds to a kth term of γ. Then, for $j < g-2$, $c_{i,j+1}$ is a color of $(k + \frac{\lambda}{2})$th term of γ. Moreover, if $j = g-2$ then $c_{i,j+1}$ is a color of $(k-1)$th term of γ. Therefore (3) is guaranteed. Similarly, if $c_{i,g-1}$ corresponds to lth term of γ then $c_{i+x,0}$ is a color of $(l+1)$th term of γ, and therefore (4) is satisfied. □

Assume that $b = \frac{n}{4}$. Since $\gcd(a, b) = 1$ and $g = \gcd(a, n) > 1$, therefore a is even, b is odd and either $g = 2$ or $g = 4$.

Lemma 17. *Let $n \equiv 4 \pmod 8$ and $g = \gcd(a, n) = 2$ or $g = 4$. A circulant $G = C(n; a, \frac{n}{4}, \frac{n}{2})$ has $\chi(G) = 4$.*

Proof. Notice that the vertices $0, \frac{n}{4}, \frac{n}{2}$ and $\frac{3n}{4}$ induce in G a clique of order 4. Hence $\chi(G) \ge 4$.

Let $b = \frac{n}{4}$. Hence b is odd and a is even. Similarly as in the above proof, denote by $H = \langle b \rangle$ the cyclic group generated by b. Clearly, H has order 4. The elements of $\mathbb{Z}_n$ can be decomposed into b cosets of the form $H + ja$, $j = 0, 1, \ldots, b-1$. Let elements in a coset $H + ja$ be ordered as follows: ja, $ja + \frac{n}{4}$, $ja + \frac{n}{2}$, $ja + \frac{3n}{4}$. Clearly, $ab \equiv 0 \pmod n$ (if $g = 4$) or $ab \equiv \frac{n}{2} \pmod n$ (if $g = 2$); thus ab is either the first or third element of the coset $H + 0a$. For each element f of $\mathbb{Z}_n$ there exists a unique couple $(i, j) \in \mathbb{Z}_4 \times \mathbb{Z}_b$ such that $f = ib + ja$, where j and i uniquely determine a coset and a position of f in $H + ja$, respectively.

Let $c_{i,j}$ denote a color of the element $ib + ja \in \mathbb{Z}_n$. A 4-coloring of G can be defined as follows. Let $c_{0,j} = B$, $c_{1,j} = G$, $c_{2,j} = R$ and $c_{3,j} = Y$, for even $j = 0, 2, \ldots, g-3$. Moreover, let $c_{0,j} = G$, $c_{1,j} = R$, $c_{2,j} = Y$ and $c_{3,j} = B$, for odd $j = 1, 3, \ldots, g-2$. Finally we put $c_{0,g-1} = Y$, $c_{1,g-1} = B$, $c_{2,g-1} = G$ and $c_{3,g-1} = R$. It is clear that $c_{i,j} \ne c_{i+1,j}$, $c_{i,j} \ne c_{i+2,j}$, $c_{i,j} \ne c_{i,j+1}$ and $c_{i,b-1} \ne c_{i+2,0}$, for each $i \in \mathbb{Z}_4$ and each $j \in \mathbb{Z}_b$. □

5. Final result

Proof of Theorem 1. Notice that for $n = 6$ the only 5-valent circulant is $G = C(6; 1, 2, 3)$ which clearly has $\chi(G) = 6$. Moreover, $\chi(C(10; 2, 4, 5)) = 5$ by Proposition 15.

Assume that $a + b = n/2$. Then $\gcd(a, n) = 1$ or $\gcd(b, n) = 1$ and the chromatic number of G is determined by Lemma 13 and Claim 3.

In the case when $a = \frac{n}{4}$ or $b = \frac{n}{4}$ we apply Lemmas 12 and 17, and possibly Claim 3.

If ($b \equiv \pm 2a \pmod{n}$ or $a \equiv \pm 2b \pmod{n}$) then either $\gcd(a, n) = 1$ or $\gcd(b, n) = 1$ or both a, b are even (and then $\frac{n}{2}$ is odd). To determine $\chi(G)$ we apply Lemma 11, Claim 3 and Proposition 15.

Notice that G is bipartite iff a, b and $\frac{n}{4}$ are odd; then immediately $\chi(G) = 2$.

If either ($b \equiv \pm 5a \pmod{n}$ or $a \equiv \pm 5b \pmod{n}$) for $n = 16, 24, 28$, or ($b \equiv \pm 6a \pmod{n}$ or $a \equiv \pm 6b \pmod{n}$) for $n = 20$, or ($b \equiv \pm 8a \pmod{n}$ or $a \equiv \pm 8b \pmod{n}$) for $n = 22, 28$, or ($b \equiv \pm 11a \pmod{n}$ or $a \equiv \pm 11b \pmod{n}$) for $n = 40$, then $g = 1$ and moreover $\gcd(a, n) = 1$ or $\gcd(b, n) = 1$. Therefore to determine $\chi(G)$ we apply Lemma 14 and Claim 3. Notice that $C(16; 1, 5, 8) \simeq C(16; 1, 3, 8)$ and $C(28; 1, 11, 14) \simeq C(28; 1, 5, 14)$.

If ($b \equiv \pm 5a \pmod{n}$ or $a \equiv \pm 5b \pmod{n}$) and $g = 2$ for $n = 26$ then we apply Proposition 15.

In the remaining cases the chromatic number is determined by Theorems 10 and 16, Claim 3 and Proposition 15. $\square$

6. Conclusion

Even though we determine the chromatic number of 5-valent circulants completely, our proof is fairly complicated. It is a desirable goal to provide a simpler proof if possible.

It does not appear feasible to determine the chromatic number of 6-valent circulants $C(n; a, b, c)$ by the method in the present paper. However, let us point out that the chromatic number of a special class of 6-valent circulants, namely of those satisfying $a + b = c$, has been completely determined in [3] and the references therein.

Acknowledgments

This research was partially carried out while the first and second authors were visiting the Department of Mathematics and Statistics, McMaster University; they would like to thank the Department for the hospitality. The second author's research was partially supported by the Slovak grant agency, Grant No. APVV-51-009605.

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